L^p-L^q MULTIPLIERS ON LOCALLY COMPACT GROUPS

RAUAN AKYLZHANOV AND MICHAEL RUZHANSKY

Abstract. In this paper we discuss the L^p-L^q boundedness of both spectral and Fourier multipliers on general locally compact separable unimodular groups G for the range $1 < p \leq 2 \leq q < \infty$. As a consequence of the established Fourier multiplier theorem we also derive a spectral multiplier theorem on general locally compact separable unimodular groups. We then apply it to obtain embedding theorems as well as time-asymptotics for the L^p-L^q norms of the heat kernels for general positive unbounded invariant operators on G. We illustrate the obtained results for sub-Laplacians on compact Lie groups and on the Heisenberg group, as well as for higher order operators. We show that our results imply the known results for L^p-L^q multipliers such as Hörmander's Fourier multiplier theorem on $\mathbb{R}^n$ or known results for Fourier multipliers on compact Lie groups. The new approach developed in this paper relies on advancing the analysis in the group von Neumann algebra and its application to the derivation of the desired multiplier theorems.

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1. Introduction

The aim of this paper is to give sufficient conditions for the L^p-L^q boundedness of Fourier and spectral multipliers on locally compact separable unimodular groups. It is known that in this case we must have $p \le q$ and two classical results are available on $\mathbb{R}^n$, namely, Hörmander's multiplier theorem [Hör60] for $1 < p \le 2 \le q < \infty$, and Lizorkin's multiplier theorem [Liz67] for $1 < p \le q < \infty$. There is a philosophical difference between these results: Hörmander's theorem does not require any regularity of the symbol and applies to p and q separated by 2, while Lizorkin theorem applies also for $1 < p \le q \le 2$ and $2 \le p \le q < \infty$ but imposes certain regularity conditions on the symbol.

In this paper we aim at proving the Hörmander type theorem expressing conditions in terms of the sharp decay property of the spectral information associated to the operator, on general locally compact separable unimodular groups based on developing a new approach relying on the analysis in the noncommutative Lorentz spaces on the group von Neumann algebra. This suggested approach seems very effective, implying as special cases known results expressed in terms of symbols, in settings when the symbolic calculus is available. The obtained results are for general Fourier multipliers, in particular also implying new results for spectral multipliers. Lizorkin type theorem in the setting of locally compact groups requires a rather substantial modification of techniques and will appear elsewhere.

The class of groups covered by our analysis is very wide. In particular, it contains abelian, compact, nilpotent groups, exponential, real algebraic or semi-simple Lie groups, solvable groups (not all of which are type I, but we do not need to assume the group to be of type I or II), and many others. As far as we are aware our results are new in all of these non-Euclidean settings.

In this paper we focus on the L^p-L^q multipliers as opposed to the L^p-multipliers for which theorems of Mihlin-Hörmander or Marcinkiewicz types provide results for both Fourier and spectral multipliers in some settings, based on the regularity of the multiplier. L^p-multipliers have been intensively studied on different kinds of groups. However, most of the results have been obtained for L^p spectral multipliers, for which a wealth of results is available: e.g. [MS94, MRS95] on Heisenberg type groups, [CM96b] on solvable Lie groups, [MT04] on nilpotent and stratified groups, to mention only very very few. L^p Fourier multiplies have also been studied but to a lesser extent due to lack of symbolic calculus that was not available until recently, e.g. Coifman and Weiss [CW71b, CW71a] on SU(2), [RW13, RW15] and then [Fis16] on compact Lie groups, or [FR14, CR16] on graded Lie groups. A characteristic feature of the L^p-L^q multipliers is that less regularity of the symbol is required. Therefore, in this paper we concentrate on the L^p-L^q multiplier theorems aiming at obtaining unifying results for general locally compact groups. We give several applications of the obtained results to questions such as embedding theorems and dispersive estimates for evolution PDEs.

The approach to the L^p-Fourier multipliers is different from the technique proposed in this paper allowing us to avoid making the assumption that the group is compact or nilpotent. In this paper we are interested in both Fourier multipliers and spectral multipliers, for the latter some L^p-L^q results being available in some special settings, see e.g. [CGM93], and also [Cow74], as well as [ANR16a] for the case of SU(2), and for the discussion of some relations between those in the group setting we can refer to [RW15] and references therein.

In the context of harmonic analysis, group von Neumann algebras are used in two different ways. In the first approach [GPJP15], the group G can be understood as the 'frequency domain' and its group von Neumann algebra $\mathrm{VN}_R(G)$ can be thought of as the 'space domain'. In this paper, we view G as 'space domain' and its group von Neumann algebra $\mathrm{VN}_R(G)$ as the 'frequency domain'.

By the combinatorial method it is possible to establish the L^p-L^q estimates for the Poisson-type semigroup $\mathcal{P}_t$ on discrete groups G [JPPP13]. Finally we note that multiplier estimates on noncommutative groups are in general considerably more delicate than those in the commutative case, recall e.g. the asymmetry problem and its resolution in [DGR00]. A link between Fourier multipliers and Lorentz spaces on group von Neumann algebras has been outlined in [AR16].

We now proceed to making a more specific description of the considered problems.

1.1. **Hörmander's theorem on locally compact groups.** To put this in context, we recall that in [Hör60, Theorem 1.11], Lars Hörmander showed that for $1 < p \leq 2 \leq q < \infty$, if the symbol $\sigma_A\colon \mathbb{R}^n \to \mathbb{C}$ of a Fourier multiplier A acting on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ satisfies the condition

$$\sup_{s>0} s \left(\int\limits_{\xi\in\mathbb{R}^n \colon |\sigma_A(\xi)|\geq s} d\xi \right)^{\frac{1}{p}-\frac{1}{q}} < +\infty, \tag{1.1}$$

then A is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Here, as usual, the Fourier multiplier A on $\mathbb{R}^n$ acts by multiplication on the Fourier transform side, i.e.

$$\widehat{Af}(\xi) = \sigma_A(\xi)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n, f \in \mathcal{S}(\mathbb{R}^n). \tag{1.2}$$

Moreover, it then follows that

$$\|A\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \lesssim \sup_{s>0} s \left(\int\limits_{\substack{\xi\in\mathbb{R}^n \\ |\sigma_A(\xi)|\geq s}} d\xi \right)^{\frac{1}{p}-\frac{1}{q}}, \quad 1 < p \leq 2 \leq q < +\infty. \tag{1.3}$$

The L^p-L^q boundedness of Fourier multipliers has been also recently investigated in the context of compact Lie groups, and we now briefly recall the result. Let G be a compact Lie group and $\widehat{G}$ its unitary dual. For $\pi \in \widehat{G}$, we write d_π for the dimension of the (unitary irreducible) representation π. In [ANR16b] the authors have shown that, for a Fourier multiplier A acting via

$$\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi)$$

by its global symbol $\sigma_A(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ we have

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G} \\ \|\sigma_A(\pi)\|_{\mathrm{op}}\geq s}} d_\pi^2 \right)^{\frac{1}{p}-\frac{1}{q}}, \quad 1 < p \leq 2 \leq q \leq \infty. \tag{1.4}$$

Here for $\pi \in \widehat{G}$, the Fourier coefficients are defined as

$$\widehat{f}(\pi) = \int_G f(x)\pi(x)^* dx,$$

and $\|\sigma_A(\pi)\|_{\mathrm{op}}$ is the operator norm of $\sigma_A(\pi)$ as the linear transformation of the representation space of $\pi \in \widehat{G}$ identified with $\mathbb{C}^{d_\pi}$. For a general development of global symbols and the corresponding global quantization of pseudo-differential operators on compact Lie groups we can refer to [RT13, RT10].

One of the results of this paper generalises both multiplier theorems (1.3) and (1.4) to the setting of general locally compact separable unimodular groups G.

By a *left Fourier multiplier in the setting of general locally compact unimodular groups we will mean left invariant operators that are measurable with respect to the right group von Neumann algebra* $\mathrm{VN}_R(G)$, see Section 2.1 for a discusson.

Thus, in Theorem 5.1 we prove the following inequality

$$\|Af\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} s \left[\int_{t\in\mathbb{R}_+ :\, \mu_t(A)\geq s} dt \right]^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p(G)}, \quad 1 < p \leq 2 \leq q < +\infty, \tag{1.5}$$

where $\mu_t(A)$ are the t-th generalised singular values of A, see Definition 2.8 for the precise definition and properties (following [TK86]). An extension of (1.5) to $q = \infty$ will be shown in Theorem 5.9.

Remark 1.1. The measurability assumption on A implies that the domain $\mathrm{Dom}(A) \subset L^2(G)$ is dense in $L^2(G)$. Without loss of generality we assume that $\mathrm{Dom}(A)$ is dense in $L^p(G)$ for every $1 < p \leq \infty$.

The proof of inequality (1.5) is based on a version of the Hausdorff-Young-Paley inequality on locally compact separable groups that we establish for this purpose.

The key idea behind the extension (1.5) is that Hörmander's theorem (1.3) can be reformulated as

$$\|A\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \lesssim \sup_{s>0} s \left(\int_{\substack{\xi\in\mathbb{R}^n \\ |\sigma_A(\xi)|\geq s}} d\xi \right)^{\frac{1}{p}-\frac{1}{q}} \simeq \|\sigma_A\|_{L^{r,\infty}(\mathbb{R}^n)} \simeq \|A\|_{L^{r,\infty}(\mathrm{VN}(\mathbb{R}^n))}, \tag{1.6}$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, $\|\sigma_A\|_{L^{r,\infty}(\mathbb{R}^n)}$ is the Lorentz space norm of the symbol σ_A, and $\|A\|_{L^{r,\infty}(\mathrm{VN}(\mathbb{R}^n))}$ is the norm of the operator A in the Lorentz space on the group von

Neumann algebra $\mathrm{VN}(\mathbb{R}^n)$ of $\mathbb{R}^n$. In turn, our estimate (1.5) is equivalent to the estimate

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \|A\|_{L^{r,\infty}(\mathrm{VN}_R(G))} \simeq \sup_{s>0} s \left[\int\limits_{t\in\mathbb{R}_+ :\ \mu_t(A)\geq s} dt \right]^{\frac{1}{r}}, \tag{1.7}$$

where A is acting on the Schwartz-Bruhat space $\mathcal{S}(G)$ and $\|A\|_{L^{r,\infty}(\mathrm{VN}_R(G))}$ is the noncommutative Lorentz space norm on the right group von Neumann algebra $\mathrm{VN}_R(G)$ of G. Thus, the Lorentz spaces become a key point for the extension of Hörmander's theorem to the setting of general locally compact (unimodular) groups.

The assumption of unimodularity of G ensures the existence of the Plancherel trace on the group von Neumann algbebra $\mathrm{VN}_R(G)$ and thus can be viewed as natural allowing one to use basic results of Fourier analysis over von Neumann algebras, such as, for example, Plancherel formula (see Segal [Seg50]). Otherwise, the noncommutative Lorenz spaces cannot be constructed as subsets of τ-measurable operators. Nevertheless, the unimodularity assumption may be in principle avoided, see e.g. [DM76], but the exposition would involve the Tomita-Takesaki modular theory and the Haagerup reduction technique. For a more detailed discussion of pseudo-differential operators in the general setting of locally compact groups (possibly non-unimodular) we refer to [MR17].

1.2. **Spectral multipliers on locally compact groups.** Let us illustrate the use of the Fourier multiplier theorem (1.5) in the important case of spectral multipliers on locally compact groups. Later, in Theorem 6.1 we will give a spectral multiplier result on general semifinite von Neumann algebras, however, we now formulate its special case for the case of group von Neumann algebras associated to locally compact groups.

Interestingly, this result asserts that the L^p-L^q norms of spectral multipliers $\varphi(|\mathcal{L}|)$ depend essentially only on the rate of growth of traces of spectral projections of the operator $|\mathcal{L}|$:

Theorem 1.2. *Let G be a locally compact separable unimodular group and let $\mathcal{L}$ be a left Fourier multiplier on G. Assume that φ is a monotonically decreasing continuous function on $[0,+\infty)$ such that*

$$\varphi(0) = 1,$$
$$\lim_{u\to+\infty} \varphi(u) = 0.$$

Then we have the inequality

$$\|\varphi(|\mathcal{L}|)\|_{L^p(G)\to L^q(G)} \lesssim \sup_{u>0} \varphi(u) \left[\tau(E_{(0,u)}(|\mathcal{L}|))\right]^{\frac{1}{p}-\frac{1}{q}}, \quad 1 < p \leq 2 \leq q < \infty. \tag{1.8}$$

Here $E_{(0,u)}(|\mathcal{L}|)$ are the spectral projections associated to the operator $|\mathcal{L}|$ to the interval $(0,u)$, see Section 2 for precise definitions, and τ is the canonical trace on the right group von Neumann algebra $\mathrm{VN}_R(G)$, see also Section 2 for a discussion.

Also we note that more general statements, weakening the above assumptions on φ, are possible, see Corollary 6.3.

The estimate (1.8) says that if the supremum on the right hand side is finite then the operator $\varphi(|\mathcal{L}|)$ is bounded from $L^p(G)$ to $L^q(G)$. Moreover, the estimate for the operator norm can be used for deriving asymptotics for propagators for equations on G. For example, we get the following consequences for the L^p-L^q norm for the heat kernel of $\mathcal{L}$, applying Theorem 1.2 with $\varphi(u) = e^{-tu}$, or embedding theorems for $\mathcal{L}$ with $\varphi(u) = \frac{1}{(1+u)^\gamma}$.

We note that estimates of the type (1.10) are exactly those leading to subsequent Strichartz estimates. Here, our method is very different from the usual ones as we do not get it by interpolation from the end-point case.

Corollary 1.3. *Let G be a locally compact unimodular separable group and let $\mathcal{L}$ be a positive left Fourier multiplier such that for some α we have*

$$\tau(E_{(0,s)}(\mathcal{L})) \lesssim s^\alpha, \quad s \to \infty. \tag{1.9}$$

Then for any $1 < p \leq 2 \leq q < \infty$ there is a constant $C = C_{\alpha,p,q} > 0$ such that we have

$$\|e^{-t\mathcal{L}}\|_{L^p(G)\to L^q(G)} \leq Ct^{-\alpha\left(\frac{1}{p}-\frac{1}{q}\right)}, \quad t > 0. \tag{1.10}$$

We also have the embeddings

$$\|f\|_{L^q(G)} \leq C\|(1+\mathcal{L})^\gamma f\|_{L^p(G)}, \tag{1.11}$$

provided that

$$\gamma \geq \alpha\left(\frac{1}{p}-\frac{1}{q}\right), \quad 1 < p \leq 2 \leq q < \infty. \tag{1.12}$$

The number α in (1.9) is determined based on the spectral properties of $\mathcal{L}$ and is often computable. For example, we have

(a) if $\mathcal{L}$ is the sub-Laplacian on a compact Lie group G then $\alpha = \frac{Q}{2}$, where Q is the Hausdorff dimension of G with respect to the control distance associated to $\mathcal{L}$;

(b) if $\mathcal{L}$ is the sub-Laplacian on the Heisenberg group $G = \mathbb{H}^n$ then $\alpha = \frac{Q}{2}$, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$;

(c) More generally, if $\mathcal{L} = (-1)^N \left(\sum\limits_{k=1}^n X_k^{2N} + \sum\limits_{k=1}^n Y_k^{2N}\right)$, where the vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n, T$ is the (usual) basis in the Lie algebra $\mathfrak{h}^n$ of the Heisenberg group $\mathbb{H}^n$ such that $[X_k, Y_k] = T$ and all other commutators are zero, then $\alpha = \frac{Q}{2N}$.

Consequently, in both of the sub-Laplacian cases (a) and (b), Corollary 1.3 implies that for any $1 < p \leq 2 \leq q < \infty$ there is a constant $C = C_{p,q} > 0$ such that we have

$$\|e^{-t\mathcal{L}}\|_{L^p(G)\to L^q(G)} \leq Ct^{-\frac{Q}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}, \quad t > 0. \tag{1.13}$$

The embeddings (1.11) under conditions (1.12) show that the *statement of Theorem 1.2 is in general sharp.* Taking $\varphi(s) = \frac{1}{(1+s)^{a/2}}$ and applying (1.8) to the sub-Laplacian Δ_{sub} in either of examples (a) or (b) above, we get that the operator $\varphi(-\Delta_{sub}) = (I - \Delta_{sub})^{-a/2}$ is $L^p(G)$-$L^q(G)$ bounded and the inequality

$$\|f\|_{L^q(G)} \leq C\|(1-\Delta_{sub})^{a/2} f\|_{L^p(G)} \tag{1.14}$$

holds true provided that

$$a \geq Q\left(\frac{1}{p} - \frac{1}{q}\right), \quad 1 < p \leq 2 \leq q < \infty. \tag{1.15}$$

However, this yields the Sobolev embedding theorem which is well-known to be sharp at least in the case (b) of the Heisenberg group ([Fol75]), showing *the sharpness of Theorem 1.2 and hence also of the Fourier multiplier theorem* (1.7). More examples are given in Section 7.

Throughout the paper we will use the notation of the type $\|f\|_X \lesssim \|f\|_Y$ if we have $\|f\|_X \leq C\|f\|_Y$ with the constant C that may depend on the spaces X, Y but not on f.

2. Notation and preliminaries

In this section we fix the notation and briefly recall some preliminaries on von Neumann algebras to be used for developing subsequent harmonic analysis on locally compact groups. For exposition purposes it seems beneficial to recall several general notions in the context of general von Neumann algebras M. However, for our application to multipliers on locally compact groups G we will be later setting M to be the right group von Neumann algebra $\mathrm{VN}_R(G)$. In particular, we will be able to readily apply the notion of noncommutative Lorentz spaces on M as developed in [H.K81], one of the key ingredients for our analysis.

Let $M \subset \mathcal{L}(\mathcal{H})$ be a semifinite von Neumann algebra acting in a Hilbert space $\mathcal{H}$ with a trace τ. The semifinite assumption simplifies the formulations and is satisfied in our main example $M = \mathrm{VN}_R(G)$.

Definition 2.1 (Affiliated operators)**.** A linear closed operator A (possibly unbounded in $\mathcal{H}$) is said to be *affiliated with* M, symbolically $A\nu M$, if it commutes with the elements of the commutant $M^!$ of M, i.e.

$$AU = UA, \quad \text{for all } \ U \in M^!. \tag{2.1}$$

This relation ν is a natural relaxation of the relation $\in$: if A is a bounded operator affiliated with M, then by the double commutant theorem $A \in M$. One of the original motivations [MVN36, MvN37] of John von Neumann was to build a mathematical foundation for quantum mechanics. In this framework, the observables with unbounded spectrum correspond to closed densely defined unbounded operators. Although the algebra M consists of bounded operators, the technique of projections makes it possible to approximate unbounded operators.

Some properties of traces shall be used in the proofs of our theorems. Therefore, we give a brief background on traces summarising the results that will be used in the sequel. For some description of measurable fields of operators and links to the representation theory and general von Neumann and C^*-algebras we refer to [FR16, Appendices B and C]. The following definition is taken from [Dix81, Definition I.6.1, p.93]:

Definition 2.2. Let M be a von Neumann algebra. A trace on the positive part $M_+ = \{A \in M \colon A^* = A > 0\}$ of M is a functional τ defined on M_+, taking non-negative, possibly infinite, real values, possessing the following properties:

- If $A \in M_+$ and $B \in M_+$, we have $\tau(A+B) = \tau(A) + \tau(B)$;
- If $A \in M_+$ and $\lambda \in \mathbb{R}_+$, we have $\tau(\lambda A) = \lambda\tau(A)$ (with the convention that $0 \cdot +\infty = 0$);
- If $A \in M_+$ and if U is a unitary operator of M, then $\tau(UAU^{-1}) = \tau(A)$.

We say that τ is faithful (or exact) if the condition $A \in M_+$, $\tau(A) = 0$, imply that $A = 0$. We say that τ is finite if $\tau(A) < +\infty$ for all $A \in M_+$. We say that τ is semifinite if, for each $A \in M_+$, $\tau(A)$ is the supremum of the numbers $\tau(B)$ over those $B \in M_+$ such that $B \leq A$ and $\tau(B) < +\infty$. We say that τ is normal if, for each increasing filtering set $\mathcal{S} \subset M_+$ with supremum $S \in M_+$, $\tau(S)$ is the supremum of $\{\tau(B)\}_{B\in\mathcal{S}}$. A von Neumann algebra M is said to be *semifinite* if there exists a semifinite faithful normal trace τ on M_+.

Definition 2.3 (τ-measurable operators $S(M)$)**.** A closeable operator A (possibly unbounded) affiliated with M is said to be *τ-measurable* if for each $\varepsilon > 0$ there exists a projection p in M such that $p\mathcal{H} \subset D(A)$ and $\tau(I-p) \leq \varepsilon$. Here $D(A)$ is the domain of A in $\mathcal{H}$. We denote by $S(M)$ the set of all τ-measurable operators.

We recall the following result which will be partially used.

Theorem 2.4 ([Seg53, Theorem 4, p. 412])**.** *If operators A and B are τ-measurable with respect to a von Neumann algebra M, then so are $A^*, A+B$ and AB, i.e. the maps*

$$+\colon M \times M \ni (A,B) \mapsto A+B \in M, \tag{2.2}$$

$$\cdot\colon M \times M \ni (A,B) \mapsto AB \in M, \tag{2.3}$$

$$*\colon M \ni A \mapsto A^* \in M \tag{2.4}$$

are well-defined.

We note that the notion of τ-measurability does not appear in the classical theory of Schatten classes since for $M = \mathcal{L}(H)$ we have $S(\mathcal{L}(H)) = \mathcal{L}(H)$.

It can be seen that $A \in M^+$ if and only if $A = (A^{1/2})^* A^{1/2}$.

Example 2.5. Let G be a locally compact unimodular group with $\mathrm{VN}_R(G)$ the group von Neumann algebra generated by the right regular representation π_R of G. Let A be a linear bounded operator commuting with the left regular representation. Then by the double commutant theorem $A \in \mathrm{VN}_R(G)$ and its action is given by

$$L^2(G) \ni h \mapsto Ah = h * K_A \in L^2(G),$$

where K_A is its convolution kernel. We can define a trace τ on $\mathrm{VN}_R^+(G)$ by

$$\tau(A) := \begin{cases} \|K_{A^{\frac{1}{2}}}\|^2_{L^2(G)}, & \text{if } K_{A^{1/2}} \in L^2(G), \\ \infty, & \text{otherwise.} \end{cases} \tag{2.5}$$

The trace τ on M_+ can also be extended to the space $S(M)$ of all τ-measurable positive operators.

Proposition 2.6. *Let (M,τ) be a von Neumann algebra and let A be a τ-measurable linear operator. Assume that φ is a Borel function on $\mathrm{sp}(|A|) \subset [0,+\infty)$. Then we* *have*

$$\tau(\varphi(|A|)) = \int_0^{+\infty} \varphi(t) d\mu(t), \tag{2.6}$$

where $\mu_t = \tau(E_t)$ *and*

$$|A| = \int_0^{+\infty} t dE_t(|A|).$$

Taking $\varphi(t) = t$, we can alternatively define a trace τ as follows

$$\tau(A) = \int t d\mu(t).$$

Proof of Proposition 2.6. For the spectral measure we can take the family $\{E_{[0,t)}\}_{t\geq 0}$ of spectral projections $E_{(0,t)}$ corresponding to the intervals $[0,t)$. The reader can check that the spectral measure axioms hold true.

The trace τ is continuous with respect to the τ-measure. In view of the monotone convergence theorem (see [TK86, Theorem 3.5]) we can assume, without loss of generality, that A is a bounded τ-measurable operator. Indeed, for every τ-measurable operator $|A|$ there exists a sequence $\{A_n\}$ of τ-measurable bounded operators

$$A_n = \int_0^n t dE_t(|A|) \leq A$$

converging to A in the τ-measure topology. Then, taking the limit

$$\lim_{n\to\infty} \tau(A_n) = \tau(A),$$

we justify the claim. We notice that every Borel function can be uniformly approximated by bounded Borel functions. Thus, we concentrate to establish (2.6) for bounded measurable A and bounded Borel functions φ on $[0, \|A\|_{B(\mathcal{H})}]$. By the spectral mapping theorem we have

$$\mathrm{sp}(\varphi(|A|)) = \varphi([0, \|A\|_{B(\mathcal{H})}]).$$

Let $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \lambda_N$ be a partition of the interval $\varphi([0, \|A\|_{B(\mathcal{H})}])$. Then the Riemann-like sums

$$R_N = \sum_{k=1}^N \lambda_k E_{\varphi^{-1}(\lambda_{k-1},\lambda_k)}(|A|)$$

converge to $\varphi(|A|)$ in the τ-measure topology. The trace τ on R_N is given by

$$\tau(R_N) = \sum_{k=1}^N \lambda_k \tau(E_{\varphi^{-1}(\lambda_{k-1},\lambda_k)}(|A|)). \tag{2.7}$$

One can notice that the sum in (2.7) is a Lebesgue integral sum

$$\sum_{k=1}^N \lambda_k \mu_{(\lambda_{k-1},\lambda_k)}$$

for the integral

$$\int_0^{\|A\|} \varphi(t) d\mu(t),$$

where we set the measure $\mu((a,b)) = \tau(E_{(a,b)}), \quad (a,b) \subset [0, \|A\|_{B(\mathcal{H})}]$. □

Example 2.7. Let $M = \{M_\varphi \colon L^2(X,\mu) \ni f \mapsto M_\varphi f = \varphi f \in L^2(X,\mu)\}_{\varphi \in L^\infty(X,\mu)}$ and take $\tau(M_\varphi) := \int_X \varphi d\mu$, where (X,μ) is a measure space. Then an operator M_φ is τ-measurable if and only if φ is a μ-almost everywhere finite function.

The $*$-algebra $S(M)$ is a basic constructon for the noncommutative integration. Let $A = U|A|$ be the polar decomposition. The spectral theorem yields that

$$|A| = \int_{\mathrm{Sp}(|A|)} \lambda dE_\lambda(|A|), \tag{2.8}$$

where $\{E_\lambda(|A|)\}_{\lambda \in \mathrm{Sp}(|A|)}$ are the spectral projections associated with the operator $|A|$. Here $dE_\lambda(|A|)$ should be understood as the relative dimension function first constructed in [MVN36]. Since A is affiliated with M, the projections satisfy $E_\lambda(|A|) \in M$. Now, we are ready to 'measure the speed of decay' of the operator A.

Definition 2.8 (Generalised t-th singular numbers)**.** For an operator $A \in S(M)$, define the distribution function $d_\lambda(A)$ by

$$d_\lambda(A) := \tau(E_{(\lambda,+\infty)}(|A|)), \quad \lambda \geq 0, \tag{2.9}$$

where $E_{(\lambda,+\infty)}(|A|)$ is the spectral projection of $|A|$ corresponding to the interval $(\lambda, +\infty)$. For any $t > 0$, we define the generalised t-th singular numbers by

$$\mu_t(A) := \inf\{\lambda \geq 0 \colon d_\lambda(A) \leq t\}. \tag{2.10}$$

For the sake of the exposition clarity we now formulate some properties of the distribution function d_A which we will be using in the proofs.

Proposition 2.9. *Let $A \in S(M)$. Then we have*

$$d_A(\mu_A(t)) \leq t; \tag{2.11}$$

$$\mu_A(t) > s \quad \textit{if and only if} \quad t < d_A(s); \tag{2.12}$$

$$\sup_{t>0} t^\alpha \mu_A(t) = \sup_{s>0} s[d_A(s)]^\alpha \quad \textit{for } \ 0 < \alpha < \infty. \tag{2.13}$$

The proof of this proposition is almost verbatim to the proof of [Gra08, Proposition 1.4.5 on page 46]. The word 'almost' stands for the right-continuity of the noncommutative distribution function $d_A(s)$ which is discussed after [TK86, Definition 1.3 on page 272]. Therefore, in the following proof we shall use the right-continuity of $d_A(s)$ without any justification.

Proof of Proposition 2.9. Let $s_n \in \{s > 0 \colon d_A(s) \leq t\}$ be such that $s_n \searrow \mu_A(t)$. Then $d_A(s_n) \leq t$ and the right-continuity of d_A implies that $d_A(\mu_A(t)) \leq t$. This proves (2.11). Now, we apply this property to derive (2.12). If $s < \mu_A(t) = \inf\{s > 0 \colon d_A(s) \leq t\}$, then s does not belong to the set $\{s > 0 \colon d_A(s) \leq t\} \implies d_A(s) > t$. Conversely, if for some t and s, we had $\mu_A(t) < s$, then the application of d_A and property (2.12) would yield the contradiciton $d_A(s) \le d_A(\mu_A(t)) \le t$. Property (2.12) is established. Finally, we show (2.13). Given $s > 0$, pick ε satisfying $0 < \varepsilon < s$. Property (2.12) yields $\mu_A(d_A(s) - \varepsilon) > s$ which implies that

$$\sup_{t>0} t^\alpha \mu_A(t) \ge (d_A(s) - \varepsilon)^\alpha \mu_A(d_A(s) - \varepsilon) > (d_A(s) - \varepsilon)^\alpha s. \tag{2.14}$$

We first let $\varepsilon \to 0$ and then take the supremum over all $s > 0$ to obtain one direction. Conversely, given $t > 0$, pick $0 < \varepsilon < \mu_A(t)$. Property (2.12) yields that $d_A(\mu_A(t) - \varepsilon) > t$. This implies that $\sup_{s>0} s(d_A(s))^\alpha \ge (\mu_A(t) - \varepsilon)(d_A(\mu_A(t) - \varepsilon))^\alpha > (\mu_A(t) - \varepsilon)t^\alpha$. We first let $\varepsilon \to 0$ and then take the supremum over all $t > 0$ to obtain the opposite direction of (2.13). ☐

Here we formulate some properties of μ_t that we use in the proof of Theorem 5.1.

Lemma 2.10 ([TK86, Lemma 2.5, p. 275])**.** *Let A, B be τ-measurable operators. Then the following properties hold true.*

(1) *The map $(0, +\infty) \ni t \mapsto \mu_t(A)$ is non-increasing and continuous from the right. Moreover,*
$$\lim_{t\to 0} \mu_t(A) = \|A\| \in [0, +\infty]. \tag{2.15}$$

(2)
$$\mu_t(A) = \mu_t(A^*). \tag{2.16}$$

(3)
$$\mu_{t+s}(AB) \le \mu_t(A)\mu_s(B). \tag{2.17}$$

(4)
$$\mu_t(ACB) \le \|A\|\|B\|\mu_t(C), \quad \textit{for any } \tau\textit{-measurable operator } C. \tag{2.18}$$

(5) *For any continuous increasing function f on $[0, +\infty)$ we have*
$$\mu_t(f(|A|)) = f(\mu_t(|A|)). \tag{2.19}$$

In Lemma 2.10, we formulate only the properties we use, whereas in [TK86, Lemma 2.5, p. 275] the reader can find more details.

Example 2.11. For the operator M_φ in Example 2.7, from Defintion 2.8 we can show its generalised t-th singular numbers to be

$$\mu_t(M_\varphi) = \varphi^*(t),$$

where $\varphi^*(t)$ is the classical decreasing rearrangement (see e.g. [BS88]).

As a noncommutative extension [H.K81] of the classical Lorentz spaces, we define Lorentz spaces $L^{p,q}(M)$ associated with a semifinite von Neumann algebra M as follows:

Definition 2.12 (Noncommutative Lorentz spaces)**.** For $1 \le p < \infty$, $1 \le q < \infty$, denote by $L^{p,q}(M)$ the set of all operators $A \in S(M)$ satisfying

$$\|A\|_{L^{p,q}(M)} := \left(\int_0^{+\infty} \left(t^{\frac{1}{p}} \mu_t(A) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < +\infty. \tag{2.20}$$

For $q = \infty$, we define $L^{p,\infty}(M)$ as the space of all operators $A \in S(M)$ satisfying

$$\|A\|_{L^{p,\infty}(M)} := \sup_{t>0} t^{\frac{1}{p}} \mu_t(A). \tag{2.21}$$

With this, for $1 \le p < \infty$, we can also define L^p-spaces on M by

$$\|A\|_{L^p(M)} := \|A\|_{L^{p,p}(M)} = \left(\int_0^{+\infty} \mu_t(A)^p \, dt \right)^{\frac{1}{p}}.$$

The classical Lorentz spaces $L^{p,q}(X, \mu)$ correspond to the case of commutative von Neumann algebra. Modulus technical details [Dix81, p. 132, Theorem 1], an arbitrary abelian von Neumann algebra in a Hilbert space $\mathcal{H}$ is isometrically isomorphic to the algebra $\{M_\varphi\}_{\varphi \in L^\infty(X,\mu)}$ from Example 2.7. Then noncommutative Lorentz spaces coincide with the classical ones:

Example 2.13 (Classical Lorentz spaces)**.** Let M be the algebra $\{M_\varphi\}_{\varphi \in L^\infty(X,\mu)}$ from Example 2.7 consisting of all the multiplication operators $M_\varphi \colon L^2(X, \mu) \ni f \mapsto M_\varphi f = \varphi f \in L^2(X, \mu)$. By Example 2.11, we have

$$\mu_t(M_\varphi) = \varphi^*(t).$$

Thus, the Lorentz space $L^{p,q}(M)$ consists of all operators M_φ such that

$$\int_0^{+\infty} [t^{\frac{1}{p}} \varphi^*(t)]^q \frac{dt}{t} < +\infty,$$

which gives the classical Lorentz space.

Concerning the structure of semifinite von Neumann algebras, given an arbitrary semifinite von Neumann algebra M with a trace τ, there is an isomorphism of M onto a certain Hilbert algebra $\mathcal{U}$ ([Dix81, p. 99, Theorem 2]). Thus, we construct the trace on the Hilbert algebra yielding the trace on M due to ismomorphism. We refer to [Dix81], [Naj72] for more details on this.

Let now G be a locally compact unimodular separable group. Denote by $\pi_L(g)$ and $\pi_R(g)$ the left and the right action of G on $L^2(G)$, respectively:

$$\pi_L(g) f(x) := f(g^{-1}x),$$
$$\pi_R(g) f(x) := f(xg),$$

and by $\mathrm{VN}_L(G)$ the group von Neumann algebra generated by all the $\pi_L(g)$ with $g \in G$, i.e.

$$\mathrm{VN}_L(G) := \{\pi_L(g)\}^{!!}_{g\in G},$$

and similary

$$\mathrm{VN}_R(G) := \{\pi_R(g)\}^{!!}_{g\in G},$$

where !! is the bicommutant of the self-adjoint subalgebras $\{\pi_L(g)\}_{g\in G}, \{\pi_R(g)\}_{g\in G} \subset \mathcal{L}(L^2(G))$. It has been shown in [Seg49] that

$$\mathrm{VN}_L(G)^! = \mathrm{VN}_R(G), \tag{2.22}$$
$$\mathrm{VN}_R(G)^! = \mathrm{VN}_L(G). \tag{2.23}$$

We do not make assumption that G is either of type I or type II. The decomposition theory for unitary representations of locally compact separable unimodular groups has been established in [Ern61, Ern62].

From now on we take $M = \mathrm{VN}_R(G)$.

For $f \in L^1(G) \cap L^2(G)$, we say that f on G *has a Fourier transform* whenever the convolution operator

$$R_f h(x) := (h * f)(x) = \int_G h(g) f(g^{-1}x)\, dg \tag{2.24}$$

is a τ-measurable operator with respect to $\mathrm{VN}_R(G)$, i.e. $R_f \in S(\mathrm{VN}_R(G))$. The Plancherel identity takes ([Seg50, Theorem 3 on page 282]) the form

$$\|R_f\|_{L^2(\mathrm{VN}_R(G))} = \|f\|_{L^2(G)}. \tag{2.25}$$

In this setting, the Hausdorff-Young inequality has been established in [Kun58] in the form

$$\|R_f\|_{L^{p'}(\mathrm{VN}_R(G))} \lesssim \|f\|_{L^p(G)}, \quad 1 < p \leq 2. \tag{2.26}$$

The constant in (2.26) has been computed in [Rus74] for simply connected real nilpotent Lie groups with explicitly computable Plancherel measures. It can be shown that the constant is less than 1 for locally compact groups with no compact open subgroups [Fou77]. In [H.K81], as an application of the technique of the t-th generalised singular values, the Hardy-Littlewood theorem ([HL27]) has been generalised to an arbitrary locally compact separable unimodular group G:

Theorem 2.14 ([H.K81])**.** *Let $1 < p \leq 2$ and $f \in L^p(G)$. Then we have*

$$\|R_f\|_{L^{p',p}(\mathrm{VN}_R(G))} \leq \|f\|_{L^p(G)}. \tag{2.27}$$

Remark 2.15. The Plancherel equality (2.25) by Segal [Seg50] and Kosaki's version [H.K81] of Hardy-Littlewood inequality (2.27) have been originally established for the left convolution $L_f h = f * h$. However, the same line of reasoning yields inequalities (2.27) and (2.25) with the right convolution R_f. We work with the right convolution operators R_f here since it naturally corresponds to left-invariant operators when analysing the Fourier multipliers on groups.

Using the technique of the t-th generalised singular values developed in [TK86], we can formulate both the Hausdorff-Young (2.26) and Hardy-Littlewood (2.27) inequalities in the following forms (for $1 < p \leq 2$):

$$\left(\int_0^{+\infty} \mu_t(R_f)^{p'} dt \right)^{\frac{1}{p'}} \equiv \|R_f\|_{L^{p'}(\mathrm{VN}_R(G))} \leq \|f\|_{L^p(G)}, \tag{2.28}$$

$$\left(\int_0^{+\infty} t^{p-2} \mu_t(R_f)^p dt \right)^{\frac{1}{p}} \equiv \|R_f\|_{L^{p',p}(\mathrm{VN}_R(G))} \leq \|f\|_{L^p(G)}. \tag{2.29}$$

In the sequel, when we prove Paley inequality in Theorem 3.1, the Hardy-Littlewood inequalities (2.27) and (2.29) (for the right convolution R_f) will also follow as its special cases.

2.1. **Fourier multipliers on locally compact groups.** Let G be a locally compact separable unimodular group. The first question is how to understand the notion of Fourier multipliers. In the first instance we adopt the following definition:

Definition 2.16. A linear operator A is said to be a left *Fourier multiplier on* G if $A \in S(\mathrm{VN}_R(G))$.

If we now recall Definition 2.1 we can see that A is a left Fourier multiplier on G if and only if A is affiliated with the right group von Neumann algebra $\mathrm{VN}_R(G)$ and is τ-measurable. We can then clarify Definition 2.16 further:

Remark 2.17. For $M = \mathrm{VN}_R(G)$ the operators affiliated with M are precisely those A that are left-invariant on G, namely,

$$A \text{ is affiliated with } \mathrm{VN}_R(G) \iff A\pi_L(g) = \pi_L(g)A, \quad \text{for all } g \in G. \tag{2.30}$$

Summarising this observation with Definition 2.16, *left Fourier multipliers on G are precisely the left-invariant operators that are measurable* (in the sense of Definition 2.3).

For clarity and in view of its importance, we give a short justification of this.

Proof of Remark 2.17. $\Longrightarrow$. By Definition 2.16, we have

$$AU = UA, \quad \text{for all } \; U \in \mathrm{VN}_R(G)^!. \tag{2.31}$$

Then by (2.23), and by taking $U = \pi_L(g)$, $g \in G$, we see that A must be left-invariant.
$\Longleftarrow$. We have

$$A\pi_L(g) = \pi_L(g)A, \quad \text{for all } \; g \in G.$$

By definition, the algebra $\mathrm{VN}_L(G)$ is the closure of the involutive subalgebra

$$\{\pi_L(g)\}_{g\in G} \subset \mathcal{L}(L^2(G))$$

in the strong operator topology. Therefore, we obtain

$$AU = UA, \quad \text{for all } \; U \in \mathrm{VN}_R(G)^!, \tag{2.32}$$

where we used (2.23). This completes the proof of Remark 2.17. $\square$

3. Paley and Hausdorff-Young-Paley inequalities

Our analysis of L^p-L^q multipliers will be based on a version of the Hausdorff-Young-Paley inequality that we establish in this section in the context of locally compact groups. It will be obtained by interpolation between the Hausdorff-Young inequality and Paley inequality that we discuss first.

We start first with an inequality that can be regarded as a Paley type inequality.

Theorem 3.1 (Paley inequality)**.** *Let G be a locally compact unimodular separable group. Let $1 < p \leq 2$. Suppose that a positive function $\varphi(t)$ satisfies the condition*

$$M_\varphi := \sup_{s>0} s \int\limits_{\substack{t\in\mathbb{R}_+ \\ \varphi(t)\geq s}} dt < +\infty. \tag{3.1}$$

Then for all $f \in L^p(G)$ we have

$$\left(\int_0^{+\infty} \mu_t(R_f)^p \varphi(t)^{2-p}\, dt\right)^{\frac{1}{p}} \leq M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G)}. \tag{3.2}$$

As usual, the integral over an empty set in (3.1) is assumed to be zero.

We note that taking $\varphi(t) = \frac{1}{t}$ we recover Kosaki's Hardy-Littlewood inequality (2.29). In this sense, the Paley inequality can be viewed as an extension of (one of) the Hardy-Littlewood inequalities. As a small byproduct of our proof of Theorem 3.1 we thus get a simple proof of Theorem 2.14.

Proof of Theorem 3.1. Let measure ν be absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^n_+$, i.e. i.e.

$$\frac{\nu(t)}{dt} := \varphi^2(t),\ t \in \mathbb{R}_+ \tag{3.3}$$

where $\frac{\nu(t)}{dt}$ is the Radon-Nikodym derivative at the point $t \in \mathbb{R}_+$. We define the corresponding space $L^p(\mathbb{R}_+, \nu)$, $1 \leq p < \infty$, as the space of complex (or real) valued functions $f = f(t)$ such that

$$\|f\|_{L^p(\mathbb{R}_+,\nu)} := \left(\int_{\mathbb{R}_+} |f(t)|^p \varphi^2(t)\, dt\right)^{\frac{1}{p}} < \infty. \tag{3.4}$$

We will show that the sub-linear operator

$$T\colon L^p(G) \ni f \mapsto Tf := \mu_t(R_f)/\varphi(t) \in L^p(\mathbb{R}_+, \nu)$$

is well-defined and bounded from $L^p(G)$ to $L^p(\mathbb{R}_+, \nu)$ for $1 < p \leq 2$. In other words, we claim that we have the estimate

$$\|Tf\|_{L^p(\mathbb{R}_+,\nu)} = \left(\int_{\mathbb{R}_+} \left(\frac{\mu_t(R_f)}{\varphi(t)}\right)^p \varphi^2(t)\, dt\right)^{\frac{1}{p}} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G)}, \tag{3.5}$$

which would give (3.2), and where we set $M_\varphi := \sup_{t>0} t \int\limits_{\substack{t\in\mathbb{R}_+ \\ \varphi(t)\geq s}} dt$. We will show that T is of weak-type (2,2) and of weak-type (1,1). More precisely, , we show that

$$\nu\{t \in \mathbb{R}_+ \colon |Tf(t)| \geq y\} \leq \left(\frac{M_2\|f\|_{L^2(G)}}{y}\right)^2 \quad \text{with norm } M_2 = 1, \tag{3.6}$$

$$\nu\{t \in \mathbb{R}_+ \colon |Tf(t)| \geq y\} \leq \frac{M_1\|f\|_{L^1(G)}}{y} \quad \text{with norm } M_1 = M_\varphi, \tag{3.7}$$

where ν is defined in (3.3). Then (3.5) would follow from (3.6) and (3.7) by the Marcinkiewicz interpolation theorem. Now, to show (3.6), using Plancherel's identity

(2.25), we get

$$y^2 \int\limits_{\substack{t\in\mathbb{R}_+\\ \frac{\mu_t(R_f)}{\varphi(t)}\geq y}} \varphi^2(t)\,dt \leq \|Tf\|^2_{L^2(\mathbb{R}_+,\nu)} = \int\limits_{\mathbb{R}_+} \left(\frac{\mu_t(R_f)}{\varphi(t)}\right)^2 \varphi^2(t)\,dt$$

$$= \int\limits_{\mathbb{R}_+} \mu_t^2(R_f) dt = \|R_f\|^2_{L^2(VN_R(G))} = \|f\|^2_{L^2(G)}.$$

Thus, T is of weak-type (2,2) with norm $M_2 \leq 1$. Further, we show that T is of weak-type (1,1) with norm $M_1 = M_\varphi$; more precisely, we show that

$$\int\limits_{\substack{t\in\mathbb{R}_+\\ \frac{\mu_t(R_f)}{\varphi(t)}\geq y}} \varphi^2(t)\,dt \lesssim M_\varphi \frac{\|f\|_{L^1(G)}}{y}. \tag{3.8}$$

From the definition of the Fourier transform it follows that

$$\mu_t(R_f) \leq \|f\|_{L^1(G)}. \tag{3.9}$$

Indeed, from the Definition 2.8, we have

$$\mu_t(R_f) \leq \|R_f\|_{L^2(G)\to L^2(G)}.$$

The Young inequality for convolution (e.g. [Fol16, p. 52, Proposition 2.39]) yields

$$\|R_f g\|_{L^2(G)} \leq \|f\|_{L^1(G)} \|g\|_{L^2(G)}.$$

Thus

$$\|R_f\|_{L^2(G)\to L^2(G)} \leq \|f\|_{L^1(G)}.$$

This proves (3.9). Therefore, we have

$$y < \frac{\mu_t(R_f)}{\varphi(t)} \leq \frac{\|f\|_{L^1(G)}}{\varphi(t)}.$$

Using this, we get

$$\left\{t \in \mathbb{R}_+ \colon \frac{\mu_t(R_f)}{\varphi(t)} > y\right\} \subset \left\{t \in \mathbb{R}_+ \colon \frac{\|f\|_{L^1(G)}}{\varphi(t)} > y\right\}$$

for any $y > 0$. Consequently,

$$\nu\left\{t \in \mathbb{R}_+ \colon \frac{\mu_t(R_f)}{\varphi(t)} > y\right\} \leq \nu\left\{t \in \mathbb{R}_+ \colon \frac{\|f\|_{L^1(G)}}{\varphi(t)} > y\right\}.$$

Setting $v := \frac{\|f\|_{L^1(G)}}{y}$, we get

$$\nu\left\{t \in \mathbb{R}_+ \colon \frac{\mu_t(R_f)}{\varphi(t)} > y\right\} \leq \int\limits_{\substack{t\in\mathbb{R}_+\\ \varphi(t)\leq v}} \varphi^2(t)\,dt. \tag{3.10}$$

We now claim that

$$\int\limits_{\substack{t\in\mathbb{R}_+\\ \varphi(t)\le v}} \varphi^2(t)\,dt \lesssim M_\varphi v. \tag{3.11}$$

Indeed, first we notice that we have

$$\int\limits_{\substack{t\in\mathbb{R}_+\\ \varphi(t)\le v}} \varphi^2(t)\,dt = \int\limits_{\substack{t\in\mathbb{R}_+\\ \varphi(t)\le v}} dt \int\limits_0^{\varphi^2(t)} d\tau.$$

We can interchange the order of integration to get

$$\int\limits_{\substack{t\in\mathbb{R}_+\\ \varphi(t)\le v}} dt \int\limits_0^{\varphi^2(t)} d\tau = \int\limits_0^{v^2} d\tau \int\limits_{\substack{t\in\mathbb{R}_+\\ \tau^{\frac{1}{2}}\le\varphi(t)\le v}} dt.$$

Further, we make a substitution $\tau = s^2$, yielding

$$\int\limits_0^{v^2} d\tau \int\limits_{\substack{t\in\mathbb{R}_+\\ \tau^{\frac{1}{2}}\le\varphi(t)\le v}} dt = 2\int\limits_0^{v} s\,ds \int\limits_{\substack{s\in\mathbb{R}_+\\ s\le\varphi(t)\le v}} dt \le 2\int\limits_0^{v} s\,ds \int\limits_{\substack{t\in\mathbb{R}_+\\ s\le\varphi(t)}} dt.$$

Since

$$s \int\limits_{\substack{t\in\mathbb{R}_+\\ s\le\varphi(t)}} dt \le \sup_{s>0} s \int\limits_{\substack{t\in\mathbb{R}_+\\ s\le\varphi(t)}} dt = M_\varphi$$

is finite by the assumption that $M_\varphi < \infty$, we have

$$2\int\limits_0^{v} s\,ds \left(\int\limits_{\substack{t\in\mathbb{R}_+\\ s\le\varphi(t)}} dt \right) \lesssim M_\varphi v.$$

This proves (3.11) and hence also (3.8). Thus, we have proved inequalities (3.6) and (3.7). Then by using the Marcinkiewicz interpolation theorem with $p_1 = 1$, $p_2 = 2$ and $\frac{1}{p} = 1 - \theta + \frac{\theta}{2}$ we now obtain

$$\left(\int\limits_{\mathbb{R}_+} \left(\frac{\mu_t(R_f)}{\varphi(t)} \right)^p \varphi^2(t)dt \right)^{\frac{1}{p}} = \|Af\|_{L^p(\mathbb{R}_+,\nu)} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G)}.$$

This completes the proof of Theorem 3.1. $\square$

Further, we recall a result on the interpolation of weighted spaces from [BL76]:

Theorem 3.2 (Interpolation of weighted spaces)**.** [BL76, 5.5.1 Theorem, p.119] *Let* $d\mu_0(x) = \omega_0(x)d\mu(x)$*,* $d\mu_1(x) = \omega_1(x)d\mu(x)$*, and write* $L^p(\omega) = L^p(\omega d\mu)$ *for the weight* ω*. Suppose that* $0 < p_0, p_1 < \infty$*. Then*

$$(L^{p_0}(\omega_0), L^{p_1}(\omega_1))_{\theta,p} = L^p(\omega),$$

where $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$*, and* $\omega = \omega_0^{p\frac{1-\theta}{p_0}} \omega_1^{p\frac{\theta}{p_1}}$*.*

From this, interpolating between the Paley-type inequality (3.2) in Theorem 3.1 and Hausdorff-Young inequality (2.28), we readily obtain an inequality that will be crucial for our subsequent analysis of L^p-L^q multipliers:

Theorem 3.3 (Hausdorff-Young-Paley inequality)**.** *Let* G *be a locally compact unimodular separable group. Let* $1 < p \le b \le p' < \infty$*. If a positive function* $\varphi(t)$*,* $t \in \mathbb{R}_+$*, satisfies condition*

$$M_\varphi := \sup_{s>0} s \int\limits_{\substack{t\in\mathbb{R}_+ \\ \varphi(t)\ge s}} dt < \infty, \tag{3.12}$$

then for all $f \in L^p(G)$ *we have*

$$\left(\int\limits_{\mathbb{R}_+} \left(\mu_t(R_f)\varphi(t)^{\frac{1}{b}-\frac{1}{p'}} \right)^b dt \right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b}-\frac{1}{p'}} \|f\|_{L^p(G)}. \tag{3.13}$$

Naturally, this reduces to the Hausdorff-Young inequality (2.28) when $b = p'$ and to the Paley inequality in (3.2) when $b = p$.

4. Nikolskii inequality on locally compact groups

In this section we establish the Nikolskii inequality (sometimes called the reverse Hölder inequality) in the setting of locally compact groups. This complements the knowledge on locally compact groups as well as gives an extension of known results on compact and on nilpotent Lie groups.

Let G be a locally compact unimodular separable group and $\mathrm{VN}_R(G)$ its right group von Neumann algebra with trace τ. We shall denote by $\mathcal{F}^R[f]$ the right Fourier transform of $f \in L^1(G)$, i.e.

$$\mathcal{F}^R[f] = R_f \colon L^2(G) \ni h \mapsto \mathcal{F}^R[f](h) = R_f[h] = h * f \in L^2(G). \tag{4.1}$$

The reason to introduce the new notation $\mathcal{F}^R[f]$ is to emphasise the connection with the classical Nikolskii inequality [Nik51].

Let us denote by $\mathrm{supp}^R(\widehat{f})$ the subspace of $L^2(G)$ orthogonal to the kernel $\mathrm{Ker}(\mathcal{F}^R[f])$ of the Fourier transform $\mathcal{F}^R[f]$, i.e.

$$\mathrm{supp}^R[\widehat{f}] := \mathrm{Ker}\left(\mathcal{F}^R[f]\right)^\perp, \tag{4.2}$$

where $\mathrm{Ker}(\mathcal{F}[f]) \subset L^2(G)$ is the kernel of the operator $\mathcal{F}^R[f]$ in (4.1).

We note that the classical Nikolskii inequality is an L^p-L^q estimate for norms of the same functions for $p < q$ so that the Fourier transforms of the functions under consideration must have bounded support.

The main question in the setting of locally compact groups is to find an analogue of the condition for bounded support of Fourier transforms since we may not have a canonical operator to use its spectral decomposition for the definition of the bounded spectrum.

Let $P_{\mathrm{supp}^R[\widehat{f}]}$ be the orthogonal projector onto the support $\mathrm{supp}^R[\widehat{f}]$. We say that $f \in L^1(G)$ has *bounded spectrum* if $\tau(P_{\mathrm{supp}^R[\widehat{f}]}) < +\infty$.

Theorem 4.1. *Let G be a locally compact separable unimodular group. Let $1 < q \leq \infty$ and $1 < p \leq \min(2, q)$. Assume that $\tau(P_{\mathrm{supp}^R[\widehat{f}]}) < \infty$, where $P_{\mathrm{supp}^R[\widehat{f}]}$ denotes the orthogonal projector onto the support $\mathrm{supp}^R[\widehat{f}]$. Then we have*

$$\|f\|_{L^q(G)} \lesssim \left(\tau(P_{\mathrm{supp}^R[\widehat{f}]})\right)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p(G)}, \tag{4.3}$$

with the constant in (4.3) independent of f.

The expression $\tau(P_{\mathrm{supp}^R[\widehat{f}]})$ can be seen as an 'eigenvalue counting function' for linear closed operators $D\colon L^2(G) \to L^2(G)$ affiliated with $\mathrm{VN}_L(G)$.

Example 4.2. Let G be a compact Lie group and let $f_L = \sum\limits_{\substack{\pi \in \widehat{G} \\ \langle \pi \rangle \leq L}} d_\pi \operatorname{Tr} \widehat{f}(\pi)\pi$, $L > 0$, be a trigonometric polynomial and $\langle \pi \rangle$ are the eigenvalues of the first-order elliptic pseudo-differential operator $(I - \Delta_G)^{\frac{1}{2}}$, i.e. $\Delta_G \pi_{ij} = \langle \pi \rangle \pi_{ij}$, $\quad i, j = \overline{1, d_\pi}$, and Δ_G is the Laplacian on G. Then we have $\tau(P_{\mathrm{supp}^R[\widehat{f_L}]}) = \sum\limits_{\substack{\pi \in \widehat{G} \\ \langle \pi \rangle \leq L}} d_\pi^2 \cong L^n$, $\quad n = \dim(G)$, where we take $D = (I - \Delta_G)^{\frac{1}{2}}$ and use the Weyl's asymptotic law (see e.g. [Shu87]) for the eigenvalue counting function.

In [NRT16, NRT15] Nikolskii inequality has been established on compact Lie groups and on compact homogeneous manifolds, respectively. We prove Theorem 4.1 along the lines of the proof in [NRT16] adapting the latter to the setting of locally compact groups. Alternatively, Example 4.2 yields another proof of the Nikolskii inequality that has been established [NRT16, NRT15] on compact Lie groups and on compact homogeneous manifolds, respectively, for functions with bounded support of the non-commutative Fourier coefficients.

In [CR16] the Nikolskii inequality was proved in the setting of graded groups (see also [CR17]): Let G be a graded Lie group of homogeneous dimension Q and let $D = \mathcal{R}$ be a positive Rockland operator of order ν. For every $L > 0$ let us consider the operator, defined by the spectral theory

$$f_L := \chi_L(\mathcal{R})[f], \tag{4.4}$$

where χ_L is the characteristic function of the interval $[0, L]$. In these notations, it was shown in [CR16, Theorem 3.1] that we have

$$\|T_L\|_{L^q(G)} \leq C L^{\frac{Q}{\nu}\left(\frac{1}{p}-\frac{1}{q}\right)} \|T_L\|_{L^p(G)}, \quad 1 \leq p \leq q \leq \infty, \tag{4.5}$$

with constant C explicitly depending on the spectral resolution of the Rockland operator $\mathcal{R}$. In other words, we have $\tau(P_{\mathrm{supp}^R[\widehat{f_L}]}) \cong L^{\frac{Q}{\nu}}$.

Proof of Theorem 4.1. We will give the proof of (4.3) in three steps. We can abbreviate $\mathcal{F}^R[f]$ in the proof to simply writing $\mathcal{F}[f]$.

Step 1. The case $p = 2$ and $q = \infty$. We have (by e.g. [Hay14, Proposition A.1.2. p. 216]) that

$$\left|\mathrm{Tr}(\widehat{f}(\pi)\pi(x))\right| \leq \mathrm{Tr}\left|\widehat{f}(\pi)\right|, \quad x \in G. \tag{4.6}$$

We notice that

$$\mathcal{F}[f]P_{\mathrm{supp}^R[\widehat{f}]} = \mathcal{F}[f].$$

Then by [TK86, Lemma 2.6, p. 277], we have

$$\mu_s(\mathcal{F}[f]) = 0, \quad s \geq \tau(P_{\mathrm{supp}^R[\widehat{f}]}). \tag{4.7}$$

From now on we shall denote $t := \tau(P_{\mathrm{supp}^R[\widehat{f}]})$ throughout the proof. Further, the application of [TK86, Proposition 2.7, p.277] yields

$$\tau(|\mathcal{F}[f]|) = \int\limits_0^\infty \mu_s(\mathcal{F}[f])\, ds = \int\limits_0^{\tau(P_{\mathrm{supp}^R[\widehat{f}]})} \mu_s(\mathcal{F}[f])\, ds, \tag{4.8}$$

where we used (4.7) in the last equality. Combining (4.8) and (4.6), we obtain

$$\begin{aligned}
\|f\|_{L^\infty(G)} &\leq \int\limits_{\widehat{G}} \mathrm{Tr}\left|\widehat{f}(\pi)\right| d\pi = \tau(|\mathcal{F}[f]|) = \int\limits_0^t \mu_s(\mathcal{F}[f]) ds \\
&\leq \left(\int\limits_0^t ds\right)^{\frac{1}{2}} \left(\int\limits_0^t \mu_s^2(\mathcal{F}[f]) ds\right)^{\frac{1}{2}} = \sqrt{\tau(P_{\mathrm{supp}^R[\widehat{f}]})}\|f\|_{L^2(G)},
\end{aligned} \tag{4.9}$$

where in the last inequality we used the Plancherel identity.

Step 2. The case $p = 2$ and $2 < q \leq \infty$. We take $1 \leq q' < 2$ so that $\frac{1}{q} + \frac{1}{q'} = 1$. We set $r := \frac{2}{q'}$ so that its dual index r' satisfies $\frac{1}{r'} = 1 - \frac{q'}{2}$. By the Hausdorff-Young inequality in (2.28), and by Hölder's inequality, we obtain

$$
\begin{aligned}
\|f\|_{L^q(G)} \leq \|\mathcal{F}[f]\|_{L^{q'}(\mathrm{VN}_R(G))} &= \left(\int_0^t \mu_s^{q'}(\mathcal{F}[f])ds\right)^{\frac{1}{q'}} \\
&\leq \left(\int_0^t ds\right)^{\frac{1}{q'r'}} \left(\int_0^t \mu_s^{q'r}(\mathcal{F}[f])\, ds\right)^{\frac{1}{q'r}} \\
&= \left(\int_0^t ds\right)^{\frac{1}{q'}-\frac{1}{2}} \left(\int_0^t \mu_s^{2}(\mathcal{F}[f])\, ds\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^t ds\right)^{\frac{1}{q'}-\frac{1}{2}} \left(\int_0^\infty \mu_s^{2}(\mathcal{F}[f])\, ds\right)^{\frac{1}{2}} \\
&= \tau(P_{\mathrm{supp}^R[\widehat{f}]})^{\frac{1}{2}-\frac{1}{q}}\|f\|_{L^2(G)},
\end{aligned}
$$

where we have used that $\frac{q'r}{2}=1$.

Step 3. If $p=\min(2,q)$ and $p\neq 2$, then $p=q$ and there is nothing to prove. For $1<p<\min(2,q)$, we claim to have

$$\|f\|_{L^q(G)} \leq \tau(P_{\mathrm{supp}^R[\widehat{f}]})^{(1/p-1/q)}\|f\|_{L^p(G)}.$$

Indeed, if $q=\infty$, for $f\not\equiv 0$, we get

$$
\begin{aligned}
\|f\|_{L^2} &= \||f|^{1-p/2}|f|^{p/2}\|_{L^2} \leq \||f|^{1-p/2}\|_{L^\infty}\||f|^{p/2}\|_{L^2} \\
&= \|f\|_{L^\infty}^{1-p/2}\||f|^{p/2}\|_{L^2} = \|f\|_{L^\infty}\|f\|_{L^\infty}^{-p/2}\||f|^{p/2}\|_{L^2} \\
&= \|f\|_{L^\infty}\|f\|_{L^\infty}^{-p/2}\|f\|_{L^p}^{p/2} \\
&\leq \tau(P_{\mathrm{supp}^R[\widehat{f}]})^{1/2}\|f\|_{L^2}\|f\|_{L^\infty}^{-p/2}\|f\|_{L^p}^{p/2},
\end{aligned}
\tag{4.10}
$$

where we have used (4.9) in the last line. Therefore, using that $f\not\equiv 0$, we have

$$\|f\|_{L^\infty} \leq \tau(P_{\mathrm{supp}^R[\widehat{f}]})^{1/p}\|f\|_{L^p}. \tag{4.11}$$

For $p<q<\infty$ we obtain

$$
\begin{aligned}
\|f\|_{L^q} &= \||f|^{1-p/q}|f|^{p/q}\|_{L^q} \leq \|f\|_{L^\infty}^{1-p/q}\|f\|_{L^p}^{p/q} \\
&\leq \tau(P_{\mathrm{supp}^R[\widehat{f}]})^{1/p(1-p/q)}\|f\|_{L^p}^{1-p/q}\|f\|_{L^p}^{p/q} = \tau(P_{\mathrm{supp}^R[\widehat{f}]})^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^p},
\end{aligned}
\tag{4.12}
$$

where we have used (4.11). ∎

5. Hörmander's multiplier theorem on locally compact groups

The statement that we prove in this section can be viewed as a locally compact groups analogue of the Hörmander L^p-L^q multiplier theorem [Hör60, p. 106, Theorem 1.11], however, because of the general setting of locally compact groups, the spectral rather than symbolic information is used. Moreover, as we will show, our statement in

Theorem 5.1 implies both the Hörmander theorem and the known results on compact Lie groups.

In the following statements, to unite the formulations, we adopt the convention that the sum or the integral over an empty set is zero, and that $0^0 = 0$.

For the formulation it is convenient to use the Schwartz-Bruhat spaces $\mathcal{S}(G)$ that have been developed by Bruhat [Bru61] as a way of doing distribution theory on locally compact groups. We briefly mention its basic properties and refer to [Bru61] for further details. The space $\mathcal{S}(G)$ is a barrelled, bornological and complete locally convex topological vector space. It is continuously and densely contained in the space $C_c(G)$ of compactly supported continuous functions. The space $\mathcal{S}(G)$ is dense in every $L^p(G)$, $1 \leq p < \infty$, which follows from the fact that $C_c(G)$ is dense in $L^p(G)$.

Theorem 5.1. *Let G be a locally compact unimodular group. Let $1 < p \leq 2 \leq q < +\infty$ and suppose that A is a linear continuous operator on the Schwartz-Bruhat space $\mathcal{S}(G)$. Then we have*

$$\|Af\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} s \left[\int\limits_{t\in\mathbb{R}_+ :\ \mu_t(A)\geq s} dt \right]^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p(G)}. \tag{5.1}$$

For $p = q = 2$ inequality (5.1) is sharp, i.e.

$$\|A\|_{L^2(G)\to L^2(G)} = \sup_{t\in\mathbb{R}_+} \mu_t(A). \tag{5.2}$$

Using the noncommutative Lorentz spaces $L^{r,\infty}$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, $p \neq q$, we can also write (5.1) as

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \|A\|_{L^{r,\infty}(VN_R(G))}. \tag{5.3}$$

Remark 5.2. As a consequence, we note that inequality (5.1) holds true also for $L^{p,\theta} - L^{q,\theta}$-Fourier multipliers:

$$\|A\|_{L^{p,\theta}(G)\to L^{q,\theta}(G)} \lesssim \|A\|_{L^{r,\infty}(\mathrm{VN}_R(G))}, \quad 1 \leq \theta < \infty. \tag{5.4}$$

Proof of Remark 5.2. Let us assume $p < 2 < q$ and fix p_0, p_1, q_0, q_1 such that

$$p_0 < p < p_1, \quad q_0 < q < q_1, \tag{5.5}$$

$$p_0 < 2 < q_0, \quad p_1 < 2 < q_1. \tag{5.6}$$

Applying inequality (5.1) for $p = p_0$, $q = q_0$ and $p = p_1$, $q = q_1$, we get

$$\|Af\|_{L^{q_i}(G)} \lesssim \sup_{s>0} s \left(\int\limits_{\substack{t\in\mathbb{R}_+ \\ \mu_t(A)\geq s}} dt \right)^{\frac{1}{p_i}-\frac{1}{q_i}} \|f\|_{L^{p_i}(G)}, \quad i = 0, 1. \tag{5.7}$$

A standard interpolation argument yields

$$\|A\|_{L^{p,\theta}(G)\to L^{q,\theta}(G)} \lesssim \|A\|^{1-\theta}_{L^{r_0,\infty}(\mathrm{VN}_R(G))} \|A\|^{\theta}_{L^{r_1,\infty}(\mathrm{VN}_R(G))}. \tag{5.8}$$

We now show that

$$\|A\|^{1-\theta}_{L^{r_0,\infty}(\mathrm{VN}_R(G))} \|A\|^{\theta}_{L^{r_1,\infty}(\mathrm{VN}_R(G))} \leq \|A\|_{L^{r,\infty}(\mathrm{VN}_R(G))},$$

where $\frac{1}{r\,i} = \frac{1}{p_i} - \frac{1}{q_i}$ and $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. Let us recall that

$$\|A\|_{L^{r_0,\infty}(\mathrm{VN}_R(G))} = \sup_{t>0} t^{\frac{1}{r_0}} \mu_t(A).$$

Direct calculations yield that

$$\left(\sup_{t>0} t^{\frac{1}{r_0}} \mu_t(A)\right)^{1-\theta} \left(\sup_{t>0} t^{\frac{1}{r_1}} \mu_t(A)\right)^{\theta} \leq \sup_{t>0} t^{\frac{1}{r}} \mu_t(A).$$

This completes the proof. □

Proof of Theorem 5.1. Since the algebra $S(\mathrm{VN}_R(G))$ of left Fourier multipliers A is closed under taking the adjoint $S(\mathrm{VN}_R(G)) \ni A \mapsto A^* \in S(\mathrm{VN}_R(G))$ (see [Seg53, Theorem 4, p. 412] or [Ter81a, Theorem 28 on p. 4]), and

$$\|A\|_{L^p(G)\to L^q(G)} = \|A^*\|_{L^{q'}(G)\to L^{p'}(G)}, \tag{5.9}$$

we may assume that $p \leq q'$, for otherwise we have $q' \leq (p')' = p$ and use (2.16) ensuring that $\mu_t(A^*) = \mu_t(A)$. When $f \in L^p(G)$, dualising the Hausdorff-Young inequality (2.28) gives, since $q' \leq 2$,

$$\|Af\|_{L^q(G)} \leq \left(\int_0^{+\infty} [\mu_t(R_{Af})]^{q'} dt\right)^{\frac{1}{q'}}. \tag{5.10}$$

By the left-invariance of A (e.g. [Ter81b, Proposition 3.1 on page 31]) we have

$$R_{Af} = AR_f, \quad f \in L^2(G).$$

By our assumptions, A and R_f are measurable with respect to $\mathrm{VN}_R(G)$. This makes it possible to apply Lemma 2.10 to obtain the estimate

$$\mu_t(R_{Af}) = \mu_t(AR_f) \leq \mu_t(A)\mu_t(R_f). \tag{5.11}$$

Thus, we obtain

$$\|Af\|_{L^q(G)} \leq \left(\int_0^{+\infty} [\mu_t(A)\mu_t(R_f)]^{q'} dt\right)^{\frac{1}{q'}}. \tag{5.12}$$

Now, we are in a position to apply the Hausdorff-Young-Paley inequality in Theorem 3.3. With $\varphi(t) = \mu_t(A)^r$ for $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, the assumptions of Theorem 3.3 are then satisfied, and since $\frac{1}{q'} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{q} = \frac{1}{r}$, we obtain

$$\left(\int_0^{+\infty} [\mu_t(R_f)\mu_t(A)]^{q'} dt\right)^{\frac{1}{q'}} \leq \sup_{s>0}\left[s \int\limits_{\substack{t\in\mathbb{R}_+ \\ \mu_t(A)^r \geq s}} dt \right]^{\frac{1}{r}} \|f\|_{L^p(G)}. \tag{5.13}$$

Further, it can be easily checked that

$$\left(\sup_{s>0} s \int\limits_{\substack{t\in\mathbb{R}_+\\ \mu_t(A)^r\geq s}} dt\right)^{\frac{1}{r}} = \left(\sup_{s>0} s^r \int\limits_{\substack{t\in\mathbb{R}_+\\ \mu_t(A)\geq s}} dt\right)^{\frac{1}{r}} = \sup_{s>0} s \left(\int\limits_{\substack{t\in\mathbb{R}_+\\ \mu_t(A)\geq s}} dt\right)^{\frac{1}{r}}. \tag{5.14}$$

Thus, we have established inequality (5.1). This completes the proof. ☐

Remark 5.3. As a special case with $G = \mathbb{R}^n$, Theorem 5.1 implies the Hörmander multiplier estimate (1.3) established in [Hör60, p. 106, Theorem 1.11], and we have

$$\|A\|_{L^{r,\infty}(\mathrm{VN}_R(\mathbb{R}^n))} = \|\sigma_A\|_{L^{r,\infty}(\mathbb{R}^n)}. \tag{5.15}$$

5.1. **The case of locally compact abelian groups.** Let G be a locally compact abelian group. The unitary dual $\widehat{G}$ consists of continuous homomorphisms $\chi\colon G \to \mathbb{C}$. Then the group von Neumann algebra is isometrically isomorphic to the multiplication algebra $L^\infty(\widehat{G})$ and the operator A acting by $\widehat{Af}(\chi) = \sigma_A(\chi)\widehat{f}(\chi)$ satisfies

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \|\sigma_A\|_{L^{r,\infty}(\widehat{G})} = \sup_{s>0} s \left(\int\limits_{\substack{\chi\in\widehat{G}\\ |\sigma_A(\chi)|\geq s}} d\chi\right)^{\frac{1}{r}},$$

where

$$\|\sigma_A\|_{L^{r,\infty}(\widehat{G})} = \sup_{t>0} t^{\frac{1}{r}} \sigma_A^*(t), \quad \frac{1}{r} = \frac{1}{p} - \frac{1}{q},\ p \neq q.$$

Here $\sigma_A^*(t)$ is a non-increasing rearrangement of the symbol $\sigma_A\colon \widehat{G} \to \mathbb{C}$.

Example 5.4. Let $G = \mathbb{T}^n$. Then we have

$$\|A\|_{L^p(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} \lesssim \sup_{s>0} s \left(\sum_{\xi\in\mathbb{Z}^n\colon |\sigma_A(\xi)|\geq s} 1\right)^{\frac{1}{p}-\frac{1}{q}}. \tag{5.16}$$

If $\sigma_A(\xi)$ tends to zero when $\xi \to \infty$, then the sum in (5.16) is always finite. In particular, we obtain from (5.16) that

$$\|A\|_{L^p(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} \lesssim \|\sigma_A\|_{\ell^\infty(\mathbb{Z}^n)}.$$

Example 5.5. Assume that $\widehat{G}$ is compact. Then Theorem 5.1 reads as follows

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} s \left(\int\limits_{\xi\in\widehat{G}\colon |\sigma_A(\xi)|\geq s} d\xi\right)^{\frac{1}{p}-\frac{1}{q}}. \tag{5.17}$$

If $\sigma_A(\xi)\colon \widehat{G} \to \mathbb{C}$ is a continuous function tending to zero at infinity, then we get

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \|\sigma_A\|_{L^\infty(\widehat{G})},$$

where we used the fact that the right-hand side in (5.17) in non-zero only for $0 < s \leq \|\sigma_A\|_{L^\infty(\widehat{G})}$. This follows from our convention that integration over empty set is zero.

In particular, for $G = \mathbb{Z}^n$ its unitary dual $\widehat{G} = \mathbb{T}^n$. The calculus of more general pseudo-difference operators have been recently developed in [BKR17].

5.2. **The case of compact Lie groups.** In this section we show that Theorem 5.1 refines the known results in the case of G being a compact Lie group. The global symbolic calculus for operators A acting on compact Lie groups has been introduced and consistently developed in [RT13, RT10], to which we refer to further details on global matrix symbols on compact Lie groups. Here we also note that with this matrix global symbol, the Fourier multiplier A must act by multiplication on the Fourier transform side

$$\widehat{Af}(\xi) = \sigma_A(\xi)\widehat{f}(\xi),\ \xi \in \widehat{G},$$

where $\widehat{f}(\xi) = \int_G f(x)\xi(x)^* dx$ is the Fourier coefficient of f at the representation $\xi \in \widehat{G}$, where for simplicity we may identify ξ with its equivalence class. As we have mentioned in (1.4), the L^p-L^q boundedness of Fourier multipliers on compact Lie groups can be controlled by its symbol $\sigma_A(\xi)$. However, Theorem 5.1 gives a better result than the known estimate (1.4); for completeness we recall the exact statement:

Theorem 5.6 ([ANR16b, ANR19])**.** *Let $1 < p \leq 2 \leq q < \infty$ and suppose that A is a Fourier multiplier on the compact Lie group G. Then we have*

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s\geq 0} s \left(\sum_{\xi\in\widehat{G}:\ \|\sigma_A(\xi)\|_{\mathrm{op}}\geq s} d_\xi^2 \right)^{\frac{1}{p}-\frac{1}{q}}, \tag{5.18}$$

where $\sigma_A(\xi) = \xi^*(g)A\xi(g)\big|_{g=e} = A\xi(e) \in \mathbb{C}^{d_\xi\times d_\xi}$ is the matrix symbol of A.

The fact that Theorem 5.1 implies Theorem 5.6 follows from the following result relating the noncommutative Lorentz norm to the global symbol of invariant operators in the context of compact Lie groups:

Proposition 5.7. *Let $1 < p \leq 2 \leq q < \infty$ and let $p \neq q$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Suppose G is a compact Lie group and A is a Fourier multiplier on G. Then we have*

$$\|A\|_{L^{r,\infty}(VN_R(G))} \lesssim \sup_{s\geq 0} s \left(\sum_{\substack{\xi\in\widehat{G} \\ \|\sigma_A(\xi)\|_{\mathrm{op}}\geq s}} d_\xi^2 \right)^{\frac{1}{p}-\frac{1}{q}}, \tag{5.19}$$

where $\sigma_A(\xi) = \xi^*(g)A\xi(g)\big|_{g=e} \in \mathbb{C}^{d_\xi\times d_\xi}$ is the matrix symbol of A.

Remark 5.8. If G is a compact Lie group, the sufficient condition (5.18) on the Fourier multiplier A implies τ-measurability of A with respect to $\mathrm{VN}_R(G)$, so we do not need to assume it explicitly in the setting of compact Lie groups. Indeed, the condition of τ-measurability does not arise in the setting of compact Lie groups due to the fact [Ter81a, Proposition 21, p. 16] that

$$A \text{ is } \tau\text{-measurable with respect to } M$$

if and only if

$$\lim_{\lambda\to+\infty} d_\lambda(A) = 0. \tag{5.20}$$

Now, if the right hand side of (5.19) is finite, the latter condition holds. Indeed, by Definition 2.12 we get

$$\begin{aligned}\sup_{s>0} s[d_s(A)]^{\frac{1}{r}} &= \sup_{t>0} t^{\frac{1}{r}}\mu_t(A)\\ &= \|A\|_{L^{r,\infty}(VN_L(G))} \leq \sup_{s>0} s \left(\sum_{\substack{\xi\in\widehat{G}\\ \|\sigma_A(\xi)\|\geq s}} d_\xi^2 \right)^{\frac{1}{p}-\frac{1}{q}} < +\infty,\end{aligned} \tag{5.21}$$

where in the first equality we used (2.13) with $\alpha = \frac{1}{r}$ from Proposition 2.9. Thus, we have

$$d_s(A) \leq \frac{C}{s^r}. \tag{5.22}$$

As a consequence, we obtain (5.20). This completes the proof.

Proof of Proposition 5.7. We first compute the norm $\|A\|_{L^{r,\infty}(\mathrm{VN}_R(G))}$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, $p \neq q$. By definition, we have

$$\|A\|_{L^{r,\infty}(\mathrm{VN}_R(G))} = \sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}}\mu_A(t). \tag{5.23}$$

The application of the property (2.13) from Proposition 2.9 yields

$$\sup_{t>0} t^{\frac{1}{r}}\mu_A(t) = \sup_{s>0} s[d_A(s)]^{\frac{1}{p}-\frac{1}{q}}.$$

Therefore, it is sufficient to show that

$$\sup_{s>0} s[d_A(s)]^{\frac{1}{p}-\frac{1}{q}} \leq \sup_{s>0} s \left(\sum_{\substack{\xi\in\widehat{G}\\ \|\sigma_A(\xi)\|\geq s}} d_\xi^2 \right)^{\frac{1}{p}-\frac{1}{q}}. \tag{5.24}$$

The latter follows from the fact that every projection is contained in its central support projection (see e.g. [Bla06, Proposition III.1.1.5.]), i.e.

$$\tau(p) \leq \tau(q) = \sum_{\substack{[\pi]\in\widehat{G}\\ \|\sigma_A(\pi)\|_{\mathrm{op}}\geq t}} d_\pi^2,$$

where q is the central support of the projection $p = E_{(t,+\infty)}(|A|)$. The proof is now complete. $\square$

5.3. **The case of $q = \infty$.** We now give a version of the multiplier theorem for $q = \infty$, as a preparation to the proof of Theorem 5.10 concerning general not necessarily invariant operators.

Theorem 5.9. *Let G be a locally compact unimodular separable group and let A be a left Fourier multipler on G. Let $1 \leq \beta \leq 2$. Then we have*

$$\|A\|_{L^\beta(G)\to L^\infty(G)} \lesssim \|A\|_{L^\beta(\mathrm{VN}_R(G))}. \tag{5.25}$$

Proof of Theorem 5.9. Since G is unimodular, we have

$$Af(g) = \int\limits_G f(u) R_A(u^{-1}g)\, du. \tag{5.26}$$

Then, by Hölder's inequality, we have

$$|Af(g)| \leq \|f\|_{L^\beta(G)} \|R_A(\cdot^{-1}g)\|_{L^{\beta'}(G)} = \|f\|_{L^\beta(G)} \|R_A\|_{L^{\beta'}(G)}, \tag{5.27}$$

where we used that the Haar measure is translation-invariant. Then, by the Hausdorff-Young inequality, we have

$$\|R_A\|_{L^{\beta'}(G)} \leq \|\mathcal{F}[R_A]\|_{L^\beta(\mathrm{VN}_R(G))} = \|A\|_{L^\beta(\mathrm{VN}_R(G))}, \quad 1 \leq \beta \leq 2, \tag{5.28}$$

where we used that the Fourier transfrom of the kernel R_A is the operator A, i.e. $\mathcal{F}[R_A] = A$. Combining inequality (5.27) and (5.28), we get

$$|Af(g)| \leq \|f\|_{L^\beta(G)} \|A\|_{L^\beta(\mathrm{VN}_R(G))}, g \in G,\ 1 \leq \beta \leq 2. \tag{5.29}$$

Taking supremum over $g \in G$ in the left-hand side of (5.29), we obtain (5.25). This completes the proof. □

5.4. **The case of non-invariant operators.** Theorem 5.1 can be extended to non-invariant operators, and also to the boundedness for non-invariant operators in Lorentz spaces, in analogy to Remark 5.2.

For the formulation it is convenient to use the Schwartz-Bruhat spaces $\mathcal{S}(G)$ that have been developed by Bruhat [Bru61] as a way of doing distribution theory on locally compact groups. We briefly mention its basic properties and refer to [Bru61] for further details. The space $\mathcal{S}(G)$ is a barrelled, bornological and complete locally convex topological vector space. It is continuously and densely contained in the space $C_c(G)$ of compactly supported continuous functions. The space $\mathcal{S}(G)$ is dense in every $L^p(G)$, $1 \leq p < \infty$, which follows from the fact that $C_c(G)$ is dense in $L^p(G)$.

Theorem 5.10. *Let G be a locally compact unimodular separable group. Let $\mathcal{D} : L^2(G) \to L^2(G)$ be a closed densely defined operator such that its inverse $\mathcal{D}^{-1}$ is measurable with respect to $\mathrm{VN}_R(G)$ and such that for some $1 < \beta \leq 2$ we have*

$$\|\mathcal{D}^{-1}\|_{L^\beta(\mathrm{VN}_R(G))} < +\infty. \tag{5.30}$$

Let A be a linear continuous operator on the Schwartz-Bruhat space $\mathcal{S}(G)$. Then for any $1 < p \leq 2 \leq q < \infty$ we have

$$\|A\|_{L^p(G) \to L^q(G)} \lesssim \left(\int\limits_G \left(\|\mathcal{D} \circ A_u\|_{L^{r,\infty}(\mathrm{VN}_R(G))} \right)^\beta du \right)^{\frac{1}{\beta}}, \tag{5.31}$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$.

Here $\{A_u\}$ is the field of operators generated by varying the Schwartz kernel K_A of A, for more details we refer to the proof of Theorem 5.10. But first we observe that choosing various $\mathcal{D}$, we get different inequalities in (5.30). Thus, before proving Theorem 5.10, we illustrate it in a few examples.

Example 5.11. Let G be a compact Lie group of dimension n and let $\mathcal{L}_G$ be the Laplace operator on G. Let us take $\mathcal{D} = (I - \mathcal{L}_G)^{\frac{n}{2}}$. By the Weyl's asymptotic law (see e.g. [Shu87]) , we get

$$\lambda_k \cong k,$$

where λ_k are the eigenvalues of $\mathcal{D}$. Then, up to constant, we obtain

$$\|\mathcal{D}^{-1}\|^{\beta}_{L^{\beta}(\mathrm{VN}_R(G))} \simeq \sum_{k=1}^{\infty} \frac{1}{k^{\beta}} < +\infty,$$

for any $\beta > 1$. Thus, condition (5.30) is satisfied.

Example 5.12. Let us take G to be the Heisenberg group $\mathbb{H}^n$ with the homogeneous dimension $Q = 2n + 2$, and let $\mathcal{L}^{sub}_{\mathbb{H}^n}$ be the canonical sub-Laplacian on $\mathbb{H}^n$. It can be computed (see (7.29)) that

$$\tau(E_{(0,s)}(-\mathcal{L}^{sub}_{\mathbb{H}^n})) = C_n s^{\frac{Q}{2}}.$$

Using this and Definition 2.8, it can be shown that

$$\mu_t((I - \mathcal{L}^{sub}_{\mathbb{H}^n})^{-\alpha}) = \frac{1}{\left(1 + t^{\frac{2}{Q}}\right)^{\alpha}}.$$

From this we obtain

$$\|(I - \mathcal{L}^{sub}_{\mathbb{H}^n})^{-\alpha}\|^{\beta}_{L^{\beta}(\mathrm{VN}_R(\mathbb{H}^n))} = \int_0^{+\infty} \frac{1}{\left(1 + t^{\frac{2}{Q}}\right)^{\alpha\beta}} dt, \tag{5.32}$$

where we used the formula

$$\tau(|A|^p) = \int_0^{+\infty} \mu_t^p(A)\, dt$$

established in [TK86, Corollary 2.8, p. 278]. The integral in (5.32) is convergent if and only if $\alpha\beta > \frac{Q}{2}$.

Proof of Theorem 5.10. Let us define

$$A_u f(g) := L_{K_A(u)} f(g) = \int_G K_A(u, gt^{-1}) f(t) dt,$$

so that $A_g f(g) = Af(g)$. For each fixed $u \in G$ the operator A_u is affiliated with $\mathrm{VN}_R(G)$. Then

$$\|Af\|_{L^q(G)} = \left(\int_G |Af(g)|^q \, dg \right)^{\frac{1}{q}} \leq \left(\int_G \sup_{u \in G} |A_u f(g)|^q \, dg \right)^{\frac{1}{q}}. \tag{5.33}$$

Now, we are in position to apply Theorem 5.9. For each fixed $u \in G$, the image $A_u f(g)$ is a function on G. The operator $A = \mathcal{D}^{-1}$ is a left Fourier multiplier on G. Then, by Theorem 5.9, we get

$$\sup_{g \in G} |\mathcal{D}^{-1} h(g)| \leq \|\mathcal{D}^{-1}\|_{L^{\beta}(\mathrm{VN}_R(G))} \|h(g)\|_{L^{\beta}(G)}, \quad 1 < \beta \leq 2.$$

From this, for functions of the form $h(g) = DA_u f(g)$, we finally obtain

$$\sup_{u\in G} |A_u f(g)| = \sup_{u\in G} |\mathcal{D}^{-1}\,\mathcal{D}A_u f(g)| \leq \|\mathcal{D}^{-1}\|_{L^\beta(\mathrm{VN}_R(G))}\|\mathcal{D}A_u f\|_{L^\beta_u(G)}.$$

Therefore, using the Minkowski integral inequality to change the order of integration, we obtain

$$\|Af\|_{L^q(G)} \lesssim \left(\int\limits_G \left(\int\limits_G |\mathcal{D}\,A_u f(g)|^\beta\,du\right)^{\frac{q}{\beta}} dg\right)^{\frac{1}{q}}$$

$$= \left[\left\|\int\limits_G |\mathcal{D}\,A_u f(g)|^\beta\,du\right\|_{L^{\frac{q}{\beta}}(G)}\right]^{\frac{1}{\beta}} \leq \left[\int\limits_G \left\||\mathcal{D}\,A_u f(g)|^\beta\right\|_{L^{\frac{q}{\beta}}(G)} du\right]^{\frac{1}{\beta}}$$

$$= \left(\int\limits_G \left(\int\limits_G |\mathcal{D}\,A_u f(g)|^q\,dg\right)^{\frac{\beta}{q}} du\right)^{\frac{1}{\beta}}$$

$$\leq \left(\int\limits_G \left(\|\mathcal{D}\,A_u\|_{L^{r,\infty}(\mathrm{VN}_R(G))}\right)^\beta du\right)^{\frac{1}{\beta}} \|f\|_{L^p(G)},$$

where the last inequality holds due to Theorem 5.1.

This completes the proof of Theorem 5.10. ☐

6. Spectral multipliers on locally compact groups

In this and next section we will give an application of Theorem 5.1 to spectral multipliers.

The classical Laplace operator $\Delta_{\mathbb{R}^n}$ is affiliated with the von Neumann algebra $\mathrm{VN}(\mathbb{R}^n) = \mathrm{VN}_L(\mathbb{R}^n) = \mathrm{VN}_R(\mathbb{R}^n)$ of all convolution operators, but is not measurable on $\mathrm{VN}(\mathbb{R}^n)$. However, the Bessel potential $(I - \Delta_{\mathbb{R}^n})^{-\frac{s}{2}}$ is measurable with respect to $\mathrm{VN}(\mathbb{R}^n)$. Therefore, one of the aims of spectral multiplier theorems is to "renormalise" operators in Hilbert space $\mathcal{H}$ making them not only measurable but also bounded. In the next theorem we first describe such a relation for general semifinite von Neumann algebras, and then in Corollary 6.2 give its application to spectral multipliers.

Theorem 6.1. *Let $\mathcal{L}$ be a closed unbouned operator affiliated with a semifinite von Neumann algebra $M \subset B(\mathcal{H})$. Assume that φ is a monotonically decreasing continuous function on $[0, +\infty)$ such that*

$$\varphi(0) = 1, \tag{6.1}$$

$$\lim_{u\to+\infty} \varphi(u) = 0. \tag{6.2}$$

Then for every $1 \leq r < \infty$ we have the equality

$$\|\varphi(|\mathcal{L}|)\|_{L^{r,\infty}(M)} = \sup_{u>0} \left(\tau(E_{(0,u)}(|\mathcal{L}|))\right)^{\frac{1}{r}} \varphi(u). \tag{6.3}$$

Let $\mathcal{L}$ be an arbitrary unbounded linear operator affiliated with (M, τ). Then Theorem 6.1 says that the function $\varphi(|\mathcal{L}|)$ is necessarily affiliated with M and $\varphi(|\mathcal{L}|) \in (M, \tau)$ if and only if the r-th power φ^r of φ grows at infinity not faster than $\frac{1}{\tau(E_{(0,u)}(|\mathcal{L}|))}$, i.e. if we have the estimate

$$\varphi(u)^r \lesssim \frac{1}{\tau(E_{(0,u)}(|\mathcal{L}|))}. \tag{6.4}$$

We now give a corollary of Theorem 6.1 for $M = \mathrm{VN}_R(G)$ being the right von Neumann algebra of a locally compact unimodular group. This is formulated in Theorem 1.2 but we recall it here for readers' convenience.

Corollary 6.2. *Let G be a locally compact unimodular separable group and let $\mathcal{L}$ be a left Fourier multiplier on G. Let φ be as in Theorem 6.1. Then we have the inequality*

$$\|\varphi(|\mathcal{L}|)\|_{L^p(G)\to L^q(G)} \lesssim \sup_{u>0} \varphi(u) \left[\tau(E_{(0,u)}(|\mathcal{L}|))\right]^{\frac{1}{p}-\frac{1}{q}}, \quad 1 < p \le 2 \le q < \infty. \tag{6.5}$$

This corollary follows immediately from combining Theorem 5.1 and Theorem 6.1 with $M = \mathrm{VN}_R(G)$, also proving Theorem 1.2.

For completeness, we give another corollary (of the proof of Theorem 6.1) without assuming that φ is monotone, continuous, and satisfies conditions (6.1)-(6.2). It is these conditions that allow us to rewrite Corollary 6.3 in the more applicable form of Corollary 6.2.

Corollary 6.3. *Let G be a locally compact unimodular separable group and let $\mathcal{L}$ be a left Fourier multiplier on G. Let φ be a Borel measurable function on the spectrum $\mathrm{Sp}(|\mathcal{L}|)$. Then we have the inequality*

$$\|\varphi(|\mathcal{L}|)\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} s[\tau(E_{(s,+\infty)}(\varphi(|\mathcal{L}|)))]^{\frac{1}{p}-\frac{1}{q}}, \quad 1 < p \le 2 \le q < \infty. \tag{6.6}$$

We will prove this corollary together with the proof of Theorem 6.1.

Proof of Theorem 6.1. By defintion

$$\|\varphi(|\mathcal{L}|)\|_{L^{r,\infty}(M)} = \sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}} \mu_t(\varphi(|\mathcal{L}|)), \quad \frac{1}{r} = \frac{1}{p} - \frac{1}{q}.$$

Using property (2.13) from Proposition 2.9, we get

$$\sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}} \mu_t(\varphi(|\mathcal{L}|)) = \sup_{s>0} s[\tau(E_{(s,+\infty)}(\varphi(|\mathcal{L}|)))]^{\frac{1}{p}-\frac{1}{q}}.$$

Hence, we have

$$\|\varphi(|\mathcal{L}|)\|_{L^{r,\infty}(M)} = \sup_{s>0} s[\tau(E_{(s,+\infty)}(\varphi(|\mathcal{L}|)))]^{\frac{1}{p}-\frac{1}{q}}. \tag{6.7}$$

Since $\mathcal{L}$ is affiliated with M the spectral projections $E_\Omega(|\mathcal{L}|)$ belong to M. Let $\langle \mathcal{L} \rangle$ be an abelian subalgebra of M generated by the spectral projectors $E_{(\lambda,+\infty)}(|\mathcal{L}|)$. Let φ be a Borel measurable function on the spectrum $\mathrm{Sp}(|\mathcal{L}|)$. Then by Borel functional calculus [Arv06, Section 2.6] it is possible to construct the operator $\varphi(|\mathcal{L}|)$. This operator is a strong limit of the spectral projections $E_\Omega(|\mathcal{L}|) \in M$. Therefore $\varphi(|\mathcal{L}|)$ is affiliated with M. The distribution function of the operator $\varphi(|\mathcal{L}|)$ is given by

$$d_s(\varphi(|\mathcal{L}|)) = \tau(E_{(s,+\infty)}(\varphi(|\mathcal{L}|))). \tag{6.8}$$

This proves Corollary 6.3.

Using [KR97, Corollary 5.6.29, p.363] and the spectral mapping theorem (see [KR97, Theorem 4.1.6]), we obtain

$$\tau(E_{(s,+\infty)}(\varphi(|\mathcal{L}|))) = \tau(E_{\varphi^{-1}(s,+\infty)}(\varphi^{-1}\circ\varphi(|\mathcal{L}|))) = \tau(E_{(0,\varphi^{-1}(s))}(|\mathcal{L}|)). \tag{6.9}$$

From the hypothesis (6.2) imposed on φ and using (6.9), we get

$$\lim_{s\to+\infty}\tau(E_{(s,+\infty)}(\varphi(|\mathcal{L}|))) = \lim_{s\to+\infty}\tau(E_{(0,\varphi^{-1}(s))}(|\mathcal{L}|)) = 0. \tag{6.10}$$

Hence, the operator $\varphi(|\mathcal{L}|)$ is τ-measurable with respect to $\mathrm{VN}_R(G)$. Combining (6.7) and (6.9), we finally obtain

$$\begin{aligned}\|\varphi(|\mathcal{L}|)\|_{L^{r,\infty}(M)} &= \sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}}\mu_t(\varphi(|\mathcal{L}|)) = \sup_{s>0} s[\tau(E_{(s,+\infty)}(\varphi(|\mathcal{L}|)))]^{\frac{1}{p}-\frac{1}{q}}\\ &= \sup_{s>0} s[\tau(E_{(0,\varphi^{-1}(s))}(|\mathcal{L}|))]^{\frac{1}{p}-\frac{1}{q}} = \sup_{u>0}\varphi(u)[\tau(E_{(0,u)}(|\mathcal{L}|))]^{\frac{1}{p}-\frac{1}{q}},\end{aligned}$$

where in the last equality we used the monotonicity of φ. This completes the proof of Theorem 6.1. ☐

7. Heat kernels and embedding theorems

In this section we show that the spectral multipliers estimate (6.2) may be also used to relate spectral properties of the operators with the time decay rates for propagators for the corresponding evolution equations. We illustrate this in the case of the heat equation, when the functional calculus and the application of Theorem 6.1 to a family of functions $\{e^{-ts}\}_{t>0}$ yield the time decay rate for the solution $u = u(t,x)$ to the heat equation

$$\partial_t u + \mathcal{L}u = 0, \quad u(0) = u_0.$$

For each $t > 0$, we apply Borel functional calculus [Arv06, Section 2.6] to get

$$u(t,x) = e^{-t\mathcal{L}}u_0. \tag{7.1}$$

One can check that $u(t,x)$ satisfies equation (7.1) and the initial condition. Then by Theorem 5.1, we get

$$\|u(t,\cdot)\|_{L^q(G)} \lesssim \|e^{-t\mathcal{L}}\|_{L^{r,\infty}(\mathrm{VN}_R(G))}\|u_0\|_{L^p(G)}, \tag{7.2}$$

reducing the L^p-L^q properties of the propagator to the time asymptotics of its non-commutative Lorentz space norm.

Corollary 7.1 (The $\mathcal{L}$-heat equation)**.** *Let G be a locally compact unimodular separable group and let $\mathcal{L}$ be an unbounded positive operator affiliated with $\mathrm{VN}_R(G)$ such that for some α we have*

$$\tau(E_{(0,s)}(\mathcal{L})) \lesssim s^{\alpha}, \quad s\to\infty. \tag{7.3}$$

Then for any $1 < p \leq 2 \leq q < \infty$ we have

$$\|e^{-t\mathcal{L}}\|_{L^p(G)\to L^q(G)} \leq C_{\alpha,p,q}t^{-\alpha\left(\frac{1}{p}-\frac{1}{q}\right)}, \quad t>0. \tag{7.4}$$

Proof of Theorem 7.1. The application of Theorem 6.1 yields

$$\|e^{-t\mathcal{L}}\|_{L^{r,\infty}(\mathrm{VN}_R(G))} = \sup_{s>0}[\tau(E_{(0,s)}(|\mathcal{L}|)]^{\frac{1}{r}} e^{-ts}, \quad \frac{1}{r} = \frac{1}{p} - \frac{1}{q}.$$

Now, using this and hypothesis (7.3), we get

$$\|e^{-t\mathcal{L}}\|_{L^{r,\infty}(\mathrm{VN}_R(G))} \lesssim \sup_{s>0} s^{\frac{\alpha}{r}} e^{-ts}.$$

The standart theorems of mathematical analysis yield that

$$\sup_{s>0} s^{\frac{\alpha}{r}} e^{-ts} = \left(\frac{\alpha}{tr}\right)^{\frac{\alpha}{r}} e^{-\frac{\alpha}{r}}. \tag{7.5}$$

Indeed, let us consider a function

$$\varphi(s) = s^{\frac{\alpha}{r}} e^{-ts}.$$

We compute its derivative

$$\varphi'(s) = s^{\frac{\alpha}{r}-1} e^{-ts} \left(\frac{\alpha}{r} - st\right).$$

The only zero is $s_0 = \frac{\alpha}{rt}$ and the derivative $\varphi'(s)$ changes its sign from positive to negative at s_0. Thus, the point s_0 is a point of maximum. This shows (7.5) and completes the proof. □

Let us now show an application of Theorem 6.1 in the case of $\varphi(s) = \frac{1}{(1+s)^\gamma}$, $s \geq 0$. It shows that for the range $1 < p \leq 2 \leq q < \infty$, the Sobolev type embedding theorems for an operator $\mathcal{L}$ depend only on the spectral behaviour of $\mathcal{L}$.

Corollary 7.2 (Embedding theorems). *Let G be a locally compact unimodular separable group and let $\mathcal{L}$ be an unbounded positive operator affiliated with $\mathrm{VN}_R(G)$ such that for some α we have*

$$\tau(E_{(0,s)}(\mathcal{L})) \lesssim s^\alpha, \quad s \to \infty. \tag{7.6}$$

Then for any $1 < p \leq 2 \leq q < \infty$ we have

$$\|f\|_{L^q(G)} \leq C\|(1+\mathcal{L})^\gamma f\|_{L^p(G)}, \tag{7.7}$$

provided that

$$\gamma \geq \alpha\left(\frac{1}{p} - \frac{1}{q}\right), \quad 1 < p \leq 2 \leq q < \infty. \tag{7.8}$$

Proof. By Theorem 6.1 with $\varphi(s) = \frac{1}{(1+s)^\gamma}$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ we have

$$\|(1+\mathcal{L})^{-\gamma}\|_{L^p(G)\to L^q(G)} \lesssim \|(1+\mathcal{L})^{-\gamma}\|_{L^{r,\infty}(\mathrm{VN}_R(G))} \lesssim \sup_{s>0} s^{\frac{\alpha}{r}}(1+s)^{-\gamma}.$$

This supremum is finite for $\gamma \geq \frac{\alpha}{r}$, giving the condition (7.8). □

Now, we illustrate Theorem 6.1 and Corollary 7.1 on a number of further examples, showing that the spectral estimate (7.3) required for the L^p-L^q estimate can be readily obtained in different situations.

In Example 7.3 below we illustrate condition (7.3) in Theorem 7.1 for the homogeneous operator $\mathrm{Op}(a)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ of order $\mu \in \mathbb{R}$.

Example 7.3. Let $a(\xi)$ be a homogeneous function of degree μ and let $\mathrm{Op}(a)$ be the linear operator given by

$$\widehat{\mathrm{Op}(a)}(\xi) = a(\xi)\widehat{f}(\xi), \quad f \in S(\mathbb{R}^n), \xi \in \mathbb{R}^n.$$

According to the general theory (see [KR97, Theorem 5.6.26, p.360] and [KR97, Corollary 5.6.29, p.363]), the spectral projection $E_{(0,s)}(|\mathrm{Op}(a)|)$ corresponds to the multiplication by $\chi_{(0,s)}(|a(\xi)|)$, where $\chi_{(0,s)}(u)$ is the characteristic function of the inteval $(0, s)$. Then the trace $\tau(E_{(0,s)}(|\mathrm{Op}(a)|))$ can be computed as follows

$$\tau(E_{(0,s)}(|\mathrm{Op}(a)|)) = \int\limits_{\substack{\mathbb{R}^n \\ |a(\xi)|\leq s}} d\xi = \int\limits_{\substack{u\in\mathbb{R}^n \\ |a(u)|\leq 1}} s^{\frac{n}{\mu}} d\,u = Cs^{\frac{n}{\mu}}, \tag{7.9}$$

where we made the substitution $\xi \to s^{\frac{1}{\mu}}u$. Hence, we get

$$\tau(E_{(0,s)}(|\mathrm{Op}(a)|) = Cs^{\frac{n}{\mu}}, \tag{7.10}$$

where $C = \int\limits_{\substack{\xi\in\mathbb{R}^n \\ |a(\xi)|\leq 1}} d\xi$. The application of Theorem 7.1 yields that if

$$C = \int\limits_{\substack{\xi\in\mathbb{R}^n \\ |a(\xi)|\leq 1}} d\xi < \infty,$$

then

$$\|e^{-t\,\mathrm{Op}(a)}\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \leq c_{\alpha\,p\,q} t^{-\frac{n}{\mu}\left(\frac{1}{p}-\frac{1}{q}\right)}, \quad 1 < p \leq 2 \leq q < \infty.$$

7.1. **Sub-Riemannian structures on compact Lie groups.** First we consider the example of sub-Laplacians on compact Lie groups in which case the number α in (7.3) can be related to the Hausdorff dimension generated by the control distance of the sub-Laplacian. Moreover, we illustrate Theorem 6.1 with examples of other functions φ than in Corollary 7.1, for example $\varphi(s) = \frac{1}{(1+s)^{\alpha/2}}$, leading to the Sobolev embedding theorems.

Example 7.4. Let $\mathcal{L} = -\Delta_{sub}$ be the sub-Laplacian on a compact Lie group G, with discrete spectrum λ_k. Then by [HK16] the trace of the spectral projections $E_{(0,s)}(\mathcal{L})$ has the following asymptotics

$$\tau(E_{(0,s)}(\mathcal{L})) \lesssim s^{\frac{Q}{2}}, \quad \text{as } s \to +\infty, \tag{7.11}$$

where Q is the Hausdorff dimension of G with respect to the control distance generated by the sub-Laplacian. Let $u(t)$ be the solution to the Δ_{sub}-heat equation

$$\frac{\partial}{\partial t}u(t,x) - \Delta_{sub}u(t,x) = 0, \quad t > 0,$$
$$u(0,x) = u_0(x), \quad u_0 \in L^p(G), \quad 1 < p \leq 2.$$

Then by Corollary 7.1, we obtain

$$\|u(t,\cdot)\|_{L^q(G)} \leq C_{n,p,q} t^{-\frac{Q}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u_0\|_{L^p(G)}, \quad 1 < p \leq 2 \leq q < +\infty. \tag{7.12}$$

Let us now take $\varphi(s) = \frac{1}{(1+s)^{a/2}}$, $s \geq 0$. Then by Theorem 6.1 the operator $\varphi(-\Delta_{sub}) = (I - \Delta_{sub})^{-a/2}$ is $L^p(G)$-$L^q(G)$ bounded and the inequality

$$\|f\|_{L^q(G)} \leq C\|(1 - \Delta_{sub})^{a/2})f\|_{L^p(G)} \tag{7.13}$$

holds true provided that

$$a \geq Q\left(\frac{1}{p} - \frac{1}{q}\right), \quad 1 < p \leq 2 \leq q < \infty. \tag{7.14}$$

Here the constant C in (7.13) is given by

$$C := \|(I - \Delta_{sub})^{-a/2}\|_{L^{r,\infty}(\mathrm{VN}_R(G))}.$$

One can always associate with Δ_{sub} a version of Sobolev spaces. Let us define

$$\|f\|_{W^{a,p}_{\Delta_{sub}}(G)} := \|(I - \Delta_{sub})^{a/2} f\|_{L^p(G)}. \tag{7.15}$$

Then the Borel functional calculus (see e.g. [Arv06]) together with (7.13)-(7.14) immediately yield

$$\|f\|_{W^{b,q}_{\Delta_{sub}}(G)} \leq C\|f\|_{W^{a,p}_{\Delta_{sub}}(G)}, \quad a - b \geq Q\left(\frac{1}{p} - \frac{1}{q}\right). \tag{7.16}$$

Each sub-Riemannian structure yields a sub-Laplacian Δ_{sub} on G . If we fix a group von Neumann algebra $\mathrm{VN}_R(G)$, then inequality (7.16) depends only on the values of the trace τ on the algebra $\mathrm{VN}_R(G)$ and not on a particular choice of a sub-Laplacian Δ_{sub}. Similarly, the Sobolev spaces $W^{a,p}_{\Delta_{sub}}(G)$ do not depend on a particular choice of a sub-Laplacian.

7.2. **Sub-Laplacian on the Heisenberg group.** Here we look at the example of the Heisenberg group determining the value of α in (7.3) for the sub-Laplacian. The interesting point here is that while the spectrum of the sub-Laplacian is continuous, Theorem 6.1 can be effectively used in this situation as well.

Example 7.5. Let $\mathcal{L}$ be the positive sub-Laplacian on the Heisenberg group $\mathbb{H}^n$ and let $Q = 2n + 2$ be the homogeneous dimension of $\mathbb{H}^n$. We claim that

$$\tau(E_{(0,s)}(\mathcal{L})) \simeq s^{Q/2}. \tag{7.17}$$

Thus, under conditions of Theorem 6.1 on φ, the spectral multiplier $\varphi(\mathcal{L})$ is τ-measurable with respect to $\mathrm{VN}_R(\mathbb{H}^n)$ and the expression on the right hand side of (6.3) takes the form

$$\|\varphi(\mathcal{L})\|_{L^{r,\infty}(\mathrm{VN}_R(\mathbb{H}^n))} \simeq \sup_{u>0} u^{\frac{Q}{2r}}\varphi(u), \quad \frac{1}{r} = \frac{1}{p} - \frac{1}{q}. \tag{7.18}$$

For example, by choosing $\varphi(u) = \frac{1}{(1+u)^{a/2}}$, $\alpha > 0$, we recover the Sobolev embedding inequalities

$$\|(I + \mathcal{L})^{b/2} f\|_{L^q(\mathbb{H}^n)} \leq C\|(I + \mathcal{L})^{a/2} f\|_{L^p(\mathbb{H}^n)}, \tag{7.19}$$

provided

$$a - b \geq Q\left(\frac{1}{p} - \frac{1}{q}\right). \tag{7.20}$$

Inequality (7.19) has been established by Folland [Fol75], and it has been extended further for Rockland operators on general graded Lie groups [FR16].

Proof of Example 7.5. By Theorem 5.1, we get

$$\|\varphi(\mathcal{L})\|_{L^p(\mathbb{H}^n)\to L^q(\mathbb{H}^n)} \lesssim \|\varphi(\mathcal{L})\|_{L^{r,\infty}(\mathrm{VN}_R(\mathbb{H}^n))}. \tag{7.21}$$

Hence it is sufficient to find the conditions on φ so that the right-hand side in (7.21) is finite. By Theorem 6.1 we have

$$\|\varphi(\mathcal{L})\|_{L^{r,\infty}(\mathrm{VN}_R(G))} = \sup_{u>0}[\tau(E_{(0,u)}(\mathcal{L}))]^{\frac{1}{r}}\varphi(u). \tag{7.22}$$

We shall now show (7.17). Since $\mathcal{L}$ is affiliated with $\mathrm{VN}_R(\mathbb{H}^n)$ it can be decomposed ([Dix81, Theorem 1 on page 187])

$$\mathcal{L} = \bigoplus_{\widehat{\mathbb{H}^n}} \int \mathcal{L}_\lambda d\nu(\lambda) \tag{7.23}$$

with respect to the center

$$C = \mathrm{VN}_R(\mathbb{H}^n) \cap \mathrm{VN}_R(\mathbb{H}^n)^!$$

of the group von Neumann algebra $\mathrm{VN}_R(\mathbb{H}^n)$. Here the collection $\{\mathcal{L}_\lambda\}_{\lambda\in\widehat{\mathbb{H}^n}}$ of the (densely defined) operators $\mathcal{L}_\lambda \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ can be interpreted as the global symbol of the operator $\mathcal{L}$, as developed in [FR16].

Hence, the spectral projections $E_{(0,s)}(\mathcal{L})$ can be decomposed

$$E_{(0,s)}(\mathcal{L}) = \bigoplus_{\widehat{\mathbb{H}^n}} \int E_{(0,s)}(\mathcal{L}_\lambda)|\lambda|^n d\lambda. \tag{7.24}$$

As a consequence [Dix81, Theorem 1 on page 225], we get

$$\tau(E_{(0,s)}(\mathcal{L})) = \int_{\widehat{\mathbb{H}^n}} \tau(E_{(0,s)}[\mathcal{L}_\lambda])|\lambda|^n d\lambda. \tag{7.25}$$

The global symbol $\mathcal{L}_\lambda \colon \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ of the sub-Laplacian $\mathcal{L}$ can be found e.g. in [FR16, Lemma 6.2.1]

$$\mathcal{L}_\lambda f(u) = -|\lambda|(\Delta_{\mathbb{R}^n} f(u) - |u|^2 f(u)), \quad f \in \mathcal{S}(\mathbb{R}^n), \quad u \in \mathbb{R}^n, \tag{7.26}$$

and is a rescaled harmonic oscillator on $\mathbb{R}^n$. It is known that for each $\lambda \in \mathbb{R}\setminus\{0\}$ the operator $\mathcal{L}_\lambda$ has purely discrete spectrum

$$\mathrm{Sp}(\mathcal{L}_\lambda) = \{s_{1,\lambda} \le s_{2,\lambda} \le \ldots \le s_{m,\lambda} \le \ldots\}.$$

Thus, we have

$$\tau(E_{(0,s)}[\mathcal{L}_\lambda]) = \sum_{\substack{k\in\mathbb{N}^n \\ s_{k,\lambda}<s}} 1. \tag{7.27}$$

The eigenvalues $s_{k,\lambda}$ are well-known and are given by

$$s_{k,\lambda} = \lambda \prod_{j=1}^{n} (2k_j + 1), \tag{7.28}$$

see e.g. [NR10]. Thus, collecting (7.25), (7.27) and (7.28), we finally obtain

$$\tau(E_{(0,s)}(\mathcal{L})) = \int\limits_{\widehat{\mathbb{H}^n}} \sum_{\substack{k\in\mathbb{N}^n\\ s_{k,\lambda}<s}} 1|\lambda|^n d\lambda \quad =$$

$$\int\limits_{\widehat{\mathbb{H}^n}} \sum_{\substack{k\in\mathbb{N}^n\\ |\lambda|\prod_{j=1}^n(2k_j+1)<s}} 1|\lambda|^n d\lambda \quad =$$

$$\sum_{k\in\mathbb{N}^n} \int\limits_{\substack{\widehat{\mathbb{H}^n}\\ |\lambda|\leq\frac{s}{\prod\limits_{j=1}^{n}\left(2k_j+1\right)}}} |\lambda|^n d\lambda \quad = \quad \frac{s^{n+1}}{n+1}\prod_{j=1}^n\sum_{k_j\in\mathbb{N}}\frac{1}{(2k_j+1)^{n+1}}.$$

Summarising, we have

$$\tau(E_{(0,s)}(\mathcal{L})) = C_n s^{\frac{Q}{2}}, \tag{7.29}$$

where we used the fact the homogeneous dimension Q of the Heiseneberg group $\mathbb{H}^n$ equals $2n+2$, i.e.

$$Q = 2n+2.$$

Finally, using (7.22) it can be seen that $\|\varphi(\mathcal{L})\|_{L^{r,\infty}(\mathrm{VN}_R(\mathbb{H}^n))}$ is finite if and only if condition (7.20) holds. $\square$

7.3. **Rockland operators on the Heisenberg group.** In this section we give another application on the Heisenberg group, to functions of operators of higher orders. As in Section 7.2, let G be the Heisenberg group $\mathbb{H}^n$ and let $\mathcal{R}$ be a Rockland operator, i.e. a homogeneous left-invariant hypoelliptic differential operator. For example, the positive sub-Laplacian is a Rockland operator but in general Rockland operators do not have to be of second order.

Thus, let $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, T$, be a basis in the Lie algebra $\mathfrak{h}^n$ of the Heisenberg group $\mathbb{H}^n$ such that $[X_k, Y_k] = T$ and all other commutators are zero. Then the operator

$$\mathcal{R} = (-1)^N \left(\sum_{k=1}^n X_k^{2N} + \sum_{k=1}^n Y_k^{2N}\right)$$

is a Rockland operator (of order $2N$), positive in the operator sense. Its symbol $\sigma_{\mathcal{R}}$ (or its infinitesimal representation) is given by

$$\sigma_{\mathcal{R}}(\pi_\lambda) = |\lambda|^N \left((-1)^N \sum_{k=1}^n \partial_{u_k}^{2N} + |u|^{2N}\right), \tag{7.30}$$

acting on the functions of the variable u in the representation space $L^2(\mathbb{R}^n)$, see e.g. [FR16, page 532], where $\pi_\lambda \in \widehat{\mathbb{H}^n}$ is the Schrödinger representation. It was also shown in [tER94, Theorem 5.1] that the enumerated eigenvalues s_m^λ, $m = (m_1, \ldots, m_n)$, of $\sigma_{\mathcal{R}}(\pi_\lambda)$ have the asymptotics given by

$$s_m^\lambda \cong |\lambda|^N \prod_{k=1}^n m_k^{2N}. \tag{7.31}$$

Thus, similarly to Section 7.2, we get

$$\tau(E_{(0,s)}(|\mathcal{R}|)) = \int\limits_{\pi_\lambda \in \widehat{\mathbb{H}^n}} \tau_\lambda(E_{(0,s)}(|\sigma_{\mathcal{R}}(\pi_\lambda)|)|d\mu(\lambda) = \int\limits_{\substack{\lambda \in \mathbb{R}^n \\ \lambda \neq 0}} |\lambda|^n d\lambda \sum_{\substack{m \in \mathbb{N}^n \\ s_m^\lambda \leq s}} 1 \cong s^{\frac{Q}{2N}}, \quad (7.32)$$

determining the value of $\alpha = \frac{Q}{2N}$ in (7.3).

7.4. **Rockland operator on graded Lie groups.** We shall apply Theorem 5.1 to functions of Rockland operators on graded Lie groups. Let G be conneted, simply connected Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $(\gamma_t)_{t>0}$ be a family of dilations on $\mathfrak{g}$ of the form

$$\gamma_t(X_i) = t^{w_i} X_i,$$

where $X_1, X_2, \cdots, X_n$ is a basis in $\mathfrak{g}$ and $w_1, w_2, \ldots, w_n$ are some positive numbers called weights. A norm $|\cdot|_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is called homogeneous if $|\gamma_t^*(\mu)|_{\mathfrak{g}^*} = t\,|\mu|_{\mathfrak{g}^*}$ for all $\mu \in \mathfrak{g}^*$ and $t > 0$. One can always construct a homogeneous norm as follows

$$|\mu|_{\mathfrak{g}^*} = \inf\{\lambda > 0\colon \|\gamma_{1/\lambda}^*(\mu)\|_{\mathfrak{g}^*} \leq 1\}. \quad (7.33)$$

A differential operator A on a graded Lie group G is defined to be a Rockland operator if it is right-invariant, homogeneous and injective in each nontrivial irreducible unitary representation. We shall use basic facts on the orbit method. With every irreducible unitary representation $\pi \in \widehat{G}$ we associate a coadjoint orbit $\mathcal{O}_\pi$.

Let $\mu \in \mathfrak{g}^*$ and let τ_μ be the radical corresponding to μ, i.e.

$$\tau_\mu = \{X \in \mathfrak{g}^*\colon \mu([X, \mathfrak{g}]) = 0\}.$$

Let $X_1, X_2, \ldots, X_n$ be the strong Malcev basis in $\mathfrak{g}$ and $\mu_1, \mu_2, \ldots, \mu_n$ be the dual basis in $\mathfrak{g}^*$. There is a G-invariant Zariski-open set $U \subset \mathfrak{g}^*$ and sets of two indices S, T partitioning $\{1, \ldots, n\}$, $n = \dim(G)$. Let $V_T = \{\mu_k\}_{k \in T}$ and $V_S = \{\mu_k\}_{k \in S}$. Coadjoint orbits in U are labelled by elements $U \cap V_T$. There is a G-invariant Zariski-open set $U \subset \mathfrak{g}^*$ and a partition

$$S \cup T = \{1, \ldots, n\}$$

leading to

$$\mathfrak{g}^* = V_S \oplus V_T.$$

Let us denote by $2k$ the dimension of the coadjoint orbit $\mathcal{O}_\pi$. Let $d^S\mu$ be the Euclidean measure on V_S normalised so that the cube $\{\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_{2k}}\}$ has volume 1. Then by [Cor04, Proposition 4.3.7] we have

$$d\mu_\pi = |\mathrm{Pf}(\mu)|^{-1}\, d^S\mu. \quad (7.34)$$

Spectra of positive Rockland operators have been investigated in [tER94]. Let $\mathcal{R}$ be a positive Rockland operator of order m. Then

$$\tau\left(E_{(0,s)}(\sigma_{\mathcal{R}}(\pi))\right) \cong \int\limits_{\substack{\mu \in \mathcal{O}_\pi \\ |\mu|_{\mathfrak{g}^*} \leq s^{\frac{1}{m}}}} d\nu_\pi(\mu), \quad (7.35)$$

where $d\nu$ is the Liouville measure on the coadjoint orbit $\mathcal{O}_\pi$. Making a substitution $\mu \mapsto \gamma^*_{s^{\frac{1}{m}}}\tilde{\mu}$, we simplify the right-hand side in (7.35) as

$$\int\limits_{\substack{\mu\in\mathcal{O}_\pi\\ |\mu|_{\mathfrak{g}^*}\le s^{\frac{1}{m}}}} d\nu_\pi(\mu) = s^{\frac{\sum\limits_{k\in T} w_k}{m}} \int\limits_{\substack{\tilde{\mu}\in\mathcal{O}_{\tilde{\pi}}\\ |\tilde{\mu}|_{\mathfrak{g}^*}\le 1}} d\tilde{\mu} = s^{\frac{\sum\limits_{k\in T} w_k}{m}} \int\limits_{\mathbb{R}^{2k}\|\mu\|_{\mathbb{R}^{2k}}\le 1} |\mathrm{Pf}(\mu)|^{-1}, d^S\mu, \tag{7.36}$$

where we used that the Jacobian J of the mapping $\mathcal{O}_\pi \ni \{\mu_k\}_{k\in T} \mapsto \tilde{\mu}_k = t^{w_k}\mu_k, k \in T$ is $J = s^{\frac{\sum\limits_{k\in T} w_k}{m}}$. Let us denote by d the degree of the Pfaffian $\mathrm{Pf}(\mu)$. Thus, we obtain

$$\tau(E_{(0,s)}(\sigma_{\mathcal{R}}(\pi)) = s^{w_T/m}\psi(\pi), \tag{7.37}$$

where we denote

$$w_T = \sum_{k\in T} w_k \tag{7.38}$$

and

$$\psi(\pi) = \int\limits_{\substack{\tilde{\mu}\in\mathcal{O}_\pi\\ |\mu|_{\mathfrak{g}^*}\le 1}} d\mu_\pi. \tag{7.39}$$

Let us denote by s^π_k the ordered eigenvalues of $\sigma_R(\pi)$. It follows from (7.37) that

$$k = (s^\pi_k)^{w_T/m}\,\psi(\pi), \tag{7.40}$$

where we used the identity $\tau(E_{(0,s^\pi_k)}) = k$. In other words, we have

$$s^\pi_k = \left(\frac{k}{\psi(\pi)}\right)^{\frac{m}{w_T}}. \tag{7.41}$$

By the reduction theory, we get

$$\tau(E_{(0,s)}(\mathcal{R})) = \int\limits_{\pi\in\widehat{G}} \tau(E_{(0,s)}(\sigma_{\mathcal{R}}(\pi)))d\pi = \int\limits_{\pi\in\widehat{G}} d\pi \sum_{\substack{k\in\mathbb{N}\\ s^\pi_k\le s}} 1. \tag{7.42}$$

Changing the order of summation and integration, we get

$$\tau(E_{(0,s)}(\mathcal{R})) = \int\limits_{\pi\in\widehat{G}} d\pi \sum_{\substack{k\in\mathbb{N}\\ s^\pi_k\le s}} 1 = \sum_{k\in\mathbb{N}} 1 \int\limits_{\substack{\pi\in\widehat{G}\\ \psi(\pi)\ge\frac{k}{s^{w_T/m}}}} d\pi$$

For a slightly different approach also with the resulting spectral multipliers theorems, we refer to [RR18].

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Rauan Akylzhanov:
School of Mathematical Sciences
Queen Mary University of London
United Kingdom
E-mail address r.akylzhanov@qmul.ac.uk

Michael Ruzhansky:
Department of Mathematics: Analysis, Logic and Discrete Mathematics
Ghent University, Belgium
and
School of Mathematical Sciences
Queen Mary University of London

United Kingdom
E-mail address Michael.Ruzhansky@ugent.be